ON GEOMETRIC REALIZATIONS OF QUANTUM MODIFIED ALGEBRAS AND THEIR CANONICAL BASES

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Abstract. A conjectural geometric construction of quantum modified algebras and their canonical basses is presented by using certain localized equivariant derived categories of double framed representation varieties of quivers. In the cases corresponding to the setting in [BLM90], the conjectural construction is verified and compatible with the one in [BLM90], which gives rise to a realization of q-Schur algebras of type **A** and their canonical bases.

Contents

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1. Introduction

Let $\dot{\mathbf{U}}$ be a symmetric quantum modified algebra of Lusztig equipped with the canonical basis $\dot{\mathbf{B}}$.

In the classical work [BLM90] of Beilinson, Lusztig and MacPherson, the derived category of the double partial flag variety is studied and it is shown that there is an algebra homomorphism from the integral form of $\dot{\mathbf{U}}$ of type $\mathbf{A}$ to the Grothendieck group of this derived category. Moreover, the image of $\dot{\mathbf{B}}$ under such a homomorphism is the set of certain simple perverse sheaves, up to shifts, in this derived category as was shown in [SV00].

In the recent work [ZH08] of Zheng, a set of endomorphism functors of certain localized equivariant derived categories of framed representation varieties is shown to satisfy the defining relations of $\dot{\mathbf{U}}$, which produces a geometric realization of the tensor products of the integrable representations of $\dot{\mathbf{U}}$ and their canonical bases.

It is well known that partial flag varieties are quotient varieties of certain open subvarieties of framed representation varieties of equivoriented quivers of type $\mathbf{A}$. Thus one may hope to obtain a natural geometric construction of the pair $(\dot{\mathbf{U}}, \dot{\mathbf{B}})$ of type $\mathbf{A}$ in the setting on which Zheng's work relied, in analogy to the geometric construction in [BLM90].

In this paper, we obtain such a construction and show that it is compatible with the one in [BLM90]. Based on this construction, we make some conjectures on the geometric realization of the pair $(\dot{\mathbf{U}}, \dot{\mathbf{B}})$ for general types.

To obtain such a construction, we consider the localized equivariant derived categories $\mathscr{D}$ of the *double* framed representation varieties attached to a quiver. In principal, the localization process kills the objects whose singular supports are disjoint from certain Steinberg-like varieties studied in [N00]. Then we define an associative convolution product, say "$\cdot$", on $\mathscr{D}$ by using the left adjoints of the localization functors and the algebraic version of the general direct image functors with compact support and general inverse image functors defined by Bernstein and Lunts ([BL94], [LO08b]).

A priori, this construction does not seem to be compatible with the one in [BLM90]. We show that this is the case, by applying results from [L91], [KS97] and [R08] on the singular supports and the resolutions of singularities of the supports of certain simple perverse sheaves on the double framed representation varieties, which turn out to be representation varieties of another quiver of type $\mathbf{A}$.

Thus we have an algebra homomorphism from the integral form of $\dot{\mathbf{U}}$ of type $\mathbf{A}$ to the Grothendieck group of $\mathscr{D}$ with the convolution product "$\cdot$" and, moreover, the image of the canonical basis $\dot{\mathbf{B}}$ under this homomorphism is the set, $\mathscr{B}_d$, of isomorphism classes of certain simple perverse sheaves in $\mathscr{D}$ up to shifts. These classes of simple perverse sheaves are part of the ones parametrizing the canonical basis of the negative half of the quantum algebra in [L93].

Now the triple $(\mathscr{D}, \cdot, \mathscr{B}_d)$ makes sense in the general setting. One may conjecture, see Conjecture 4.14, that similar statements hold in the general cases, based on the study in the type $\mathbf{A}$ cases. It will be very interesting to relate the construction in the general cases to that in affine type $\mathbf{A}$ cases in [GV93] and [L99], the one in type $\mathbf{D}$ cases in [Li10a], to the geometric realization of quantum affine algebras in [N00] (in view of [T87] and [G94]), and to the categorification of $\dot{\mathbf{U}}$ in [KL08a]-[KL08c], [R08] and [MSV10].

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2. Preliminaries, I

We shall recall the definition of a symmetric quantum modified algebra from [L93].

2.1. **Symmetric Cartan data, root data and graphs.** Let I be a finite set, and

$$\cdot : \mathbb{Z}[I] \times \mathbb{Z}[I] \to \mathbb{Z}$$

be a symmetric bilinear form. The pair $(I, \cdot)$ is called a *symmetric Cartan datum* if

$$i \cdot i = 2 \quad \text{and} \quad i \cdot j \in \{0, -1, -2, \cdots\}, \quad \forall i \neq j \in I.$$

We call a triple $(\mathbf{X}, \mathbf{Y}, (,))$ a *root datum* of $(I, \cdot)$ if the following properties are satisfied:

- $\mathbf{X}$ and $\mathbf{Y}$ are finitely generated free abelian groups and $(,) : \mathbf{Y} \times \mathbf{X} \to \mathbb{Z}$ is a perfect pairing;
- there are two embeddings $I \hookrightarrow \mathbf{X}$, $i \mapsto \alpha_i$ and $I \hookrightarrow \mathbf{Y}$, $i \mapsto \check{\alpha}_i$ such that $(\check{\alpha}_i, \alpha_j) = i \cdot j$ for any $i, j \in I$.

Let $\mathbf{X}^+ = \{\lambda \in \mathbf{X} | (\check{\alpha}_i, \lambda) \in \mathbb{N}, \forall i \in I\}$ be the set of all dominant integral weights.

To a symmetric Cartan datum, associated a *graph* Γ consisting of the following data:

$$\Gamma = (I, H; {}', {}'' : H \to I; {}^- : H \to H)$$

where I is the vertex set and H is the edge set, the maps $'$ and $''$ are the source and target maps, respectively, and the map $^-$ is the fixed-point-free involution such that

$$h' \neq h'', h' = (\bar{h})'' \text{ and } \#\{h \in H | h' = i, h'' = j\} = -i \cdot j, \quad \forall h \in H, i \neq j \in I.$$

2.2. **Quantum modified algebras.** Let $\mathbb{Q}(v)$ be the rational field with the indeterminate v. Let

$$[[s]] = \frac{v^s - v^{-s}}{v - v^{-1}}, \quad [[s]]^! = [[s]][[s-1]] \cdots [[1]], \quad \text{for any } s \in \mathbb{N}.$$

The *quantum modified algebra* $\dot{\mathbf{U}}$ of Lusztig attached to a root datum $(\mathbf{X}, \mathbf{Y}, (,))$ of a symmetric Cartan datum $(I, \cdot)$ is a $\mathbb{Q}(v)$-algebra without unit determined by the following generators and relations. The generators are

$$1_\lambda, \ E_{\lambda+\alpha_i,\lambda} \text{ and } F_{\lambda-\alpha_i,\lambda}, \quad \forall i \in I, \lambda \in \mathbf{X}.$$

We set

$$E^{(n)}_{\lambda+n\alpha_i,\lambda} = \frac{1}{[[n]]^!} E_{\lambda+n\alpha_i,\lambda+(n-1)\alpha_i} \cdots E_{\lambda+2\alpha_i,\lambda+\alpha_i} E_{\lambda+\alpha_i,\lambda},$$

$$F^{(n)}_{\lambda-n\alpha_i,\lambda} = \frac{1}{[[n]]^!} F_{\lambda-n\alpha_i,\lambda-(n-1)\alpha_i} \cdots F_{\lambda-2\alpha_i,\lambda-\alpha_i} F_{\lambda-\alpha_i,\lambda}, \quad \forall n \in \mathbb{N}, i \in I, \text{ and } \lambda \in \mathbf{X}.$$

The relations are

$$(\dot{\mathbf{U}}\text{a}) \qquad 1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'} 1_\lambda, \quad \forall \lambda, \lambda' \in \mathbf{X};$$

$$(\dot{\mathbf{U}}\text{b}) \qquad E_{\lambda+\alpha_i,\lambda} 1_{\lambda'} = \delta_{\lambda,\lambda'} E_{\lambda+\alpha_i,\lambda}, \quad 1_{\lambda'} E_{\lambda+\alpha_i,\lambda} = \delta_{\lambda',\lambda+\alpha_i} E_{\lambda+\alpha_i,\lambda}, \quad \forall i \in I, \lambda, \lambda' \in \mathbf{X};$$

$$(\dot{\mathbf{U}}\text{c}) \qquad F_{\lambda-\alpha_i,\lambda} 1_{\lambda'} = \delta_{\lambda,\lambda'} F_{\lambda-\alpha_i,\lambda}, \quad 1_{\lambda'} F_{\lambda-\alpha_i,\lambda} = \delta_{\lambda',\lambda-\alpha_i} F_{\lambda-\alpha_i,\lambda}, \quad \forall i \in I, \lambda, \lambda' \in \mathbf{X};$$

$$(\dot{\mathbf{U}}\text{d}) \qquad E_{\lambda-\alpha_j+\alpha_i,\lambda-\alpha_j} F_{\lambda-\alpha_j,\lambda} - F_{\lambda+\alpha_i-\alpha_j,\lambda+\alpha_i} E_{\lambda+\alpha_i,\lambda} = \delta_{ij} [[(\check{\alpha}_i, \lambda)]] 1_\lambda, \quad \forall i, j \in I, \lambda \in \mathbf{X};$$

$$(\dot{\mathbf{U}}\text{e}) \qquad \sum_{p=0}^{m} (-1)^p E^{(m-p)}_{\lambda+m\alpha_i+\alpha_j,\lambda+p\alpha_i+\alpha_j} E_{\lambda+p\alpha_i+\alpha_j,\lambda+p\alpha_i} E^{(p)}_{\lambda+p\alpha_i,\lambda} = 0, \quad \forall i \neq j \in I, \lambda \in \mathbf{X};$$

$$(\dot{\mathbf{U}}\text{f}) \qquad \sum_{p=0}^{m} (-1)^p F^{(m-p)}_{\lambda-m\alpha_i-\alpha_j,\lambda-p\alpha_i-\alpha_j} F_{\lambda-p\alpha_i-\alpha_j,\lambda-p\alpha_i} F^{(p)}_{\lambda-p\alpha_i,\lambda} = 0, \quad \forall i \neq j \in I, \lambda \in \mathbf{X};$$

where we set $m = 1 - i \cdot j$ in the q-Serre relations ($\dot{\mathbf{U}}$e) and ($\dot{\mathbf{U}}$f).

Let $\mathbb{A} = \mathbb{Z}[v, v^{-1}]$ be the subring of Laurent polynomials in $\mathbb{Q}(v)$. Let ${}_{\mathbb{A}}\dot{\mathbf{U}}$ be the $\mathbb{A}$-subalgebra of $\dot{\mathbf{U}}$ generated by the elements 1_λ, $E^{(n)}_{\lambda+n\alpha_i,\lambda}$ and $F^{(n)}_{\lambda-n\alpha_i,\lambda}$ for all $\lambda \in \mathbf{X}$, $n \in \mathbb{N}$ and $i \in I$. Let $\dot{\mathbf{B}}$ be the canonical basis of $\dot{\mathbf{U}}$ defined in [L93, 25.2.4].

3. Preliminaries, II

We shall recall the equivariant derived category from [BL94], [LMB00], [LO08a]-[LO08b]. We will use the presentations in [S08] and [WW09].

3.1. **Derived categories $\mathcal{D}^*(X)$.** Let p be a prime number. Fix an algebraic closure k of the finite field $\mathbb{F}_p$ of p elements. All algebraic varieties in this paper will be over k.

Let l be a prime number different from p. Let $\bar{\mathbb{Q}}_l$ be an algebraic closure of the field of l-adic numbers.

We write $\mathcal{D}^b(X)$ for the *bounded derived category* of complexes of $\bar{\mathbb{Q}}_l$-constructible sheaves on an algebraic variety X defined in [BBD82, 2.2.18]. See also [FK88, I, §12] and [KW01, II 5]. We denote $\mathcal{D}^*(X)$, $* = \{\phi, +, -\}$, for the similar derived category with the term "bounded" replaced by "unbounded", "bounded below", "bounded above", respectively.

The shift functor will be denoted by "$[-]$". The functors Rf_*, $Rf_!$, Lf^*, and $Rf^!$ associated to a given morphism $f : Y \to X$ of varieties will be written as f_*, $f_!$, f^* and $f^!$, respectively, in this paper. We also write $K \to L \to M \to$ for any distinguished triangle $K \to L \to M \to K[1]$ in the derived categories just defined.

3.2. **Equivariant derived categories $\mathcal{D}^*_G(X)$.** Let us recall the equivariant derived categories following [BL94], [S08], and [WW09].

A morphism $f : Y \to X$ of varieties over k is called *n-acyclic* if for any sheaf $\mathcal{F}$ over X, considered as a complex concentrated on the zeroth degree, then

$$\mathcal{F} \simeq \tau_{\leq n} f_* f^*(\mathcal{F}), \tag{1}$$

and the property (1) holds under base changes, where $\tau_{\leq n} : \mathcal{D}(X) \to \mathcal{D}^{\leq n}(X)$ is the truncation functor.

Let G be a linear algebraic group over k.

A G-equivariant morphism $E \xrightarrow{e} X$ is called a *resolution of* X (with respect to G) if the action of G on E is free. We thus have the following diagram

$$X \xleftarrow{\ e\ } E \xrightarrow{\ \bar{e}\ } \bar{E} := G\backslash E,$$

where $G\backslash E$ is the quotient of E by G. Let $[G\backslash X]$ be the category whose objects are all resolutions of X and morphisms between resolutions of X are defined to be morphisms over X. By [LMB00, 18.7.5], $[G\backslash X]$ is an algebraic stack of Bernstein-Lunts.

A resolution $E \xrightarrow{e} X$ of X is called *smooth* if e is smooth. If, moreover, $\bar{e}$ is n-acyclic, we call the resolution $E \xrightarrow{e} X$ an *n-acyclic smooth resolution of* X.

A sequence, $(E_n, f_n)_{n\in\mathbb{N}}$, of smooth resolutions of X:

$$E_0 \xrightarrow{\ f_0\ } E_1 \xrightarrow{\ f_1\ } \cdots \longrightarrow E_n \xrightarrow{\ f_n\ } E_{n+1} \longrightarrow \cdots,$$

is called *acyclic* if $e_n : E_n \to X$ is n-acyclic and f_n is a closed inclusion, $\forall n \in \mathbb{N}$.

Given any smooth resolution $E \xrightarrow{e} X$, the *category* $\mathcal{D}^+(\bar{E}|e)$ of *objects from* X is the full subcategory of $\mathcal{D}^+(\bar{E})$ consisting of all objects K such that there is a complex $L \in \mathcal{D}^+(X)$ satisfying $e^*(L) \simeq \bar{e}^*(K)$.

If $f : E \to E'$ is a morphism of smooth resolutions of X, then it induces a map $\bar{f} : \bar{E} \to \bar{E}'$, which gives rise to the inverse image functor $\bar{f}^* : \mathcal{D}^+(\bar{E}'|e') \to \mathcal{D}^+(\bar{E}|e)$.

Given any acyclic sequence, $(E_n, f_n)_{n\in\mathbb{N}}$, of smooth resolutions of X, we then have a sequence of functors:

$$\mathcal{D}^+(\bar{E}_0|e_0) \xleftarrow{\ \bar{f}_0^*\ } \mathcal{D}^+(\bar{E}_1|e_1) \xleftarrow{\ \bar{f}_1^*\ } \cdots \mathcal{D}^+(\bar{E}_n|e_n) \xleftarrow{\ \bar{f}_n^*\ } \mathcal{D}^+(\bar{E}_{n+1}|e_{n+1}) \longleftarrow \cdots.$$

The *inverse limit category* $\varprojlim \mathcal{D}^+(\bar{E}_n|e_n)$ is defined to be the category whose objects are pairs $(A_n; \phi_n)_{n\in\mathbb{N}}$, with $A_n \in \mathcal{D}^+(\bar{E}_n|e_n)$ and $\phi_n : \bar{f}_{n+1}^*(A_{n+1}) \tilde{\to} A_n$ isomorphisms; and whose morphisms $\alpha : (A_n; \phi_n)_{n\in\mathbb{N}} \to (B_n; \psi_n)_{n\in\mathbb{N}}$ are collections of morphisms $\alpha_n : A_n \to B_n$ such that $\alpha_n \phi_n = \psi_n \bar{f}_{n+1}^*(\alpha_{n+1})$, $\forall n \in \mathbb{N}$.

Definition 3.3. The *bounded below G-equivariant derived category of* X is defined to be

$$\mathcal{D}_G^+(X) = \varprojlim \mathcal{D}^+(\bar{E}_n|e_n).$$

Similarly, we define the *bounded* (resp. *bounded above*; *unbounded*) *G-equivariant derived category* $\mathcal{D}_G^b(X)$ (resp. $\mathcal{D}_G^-(X)$; $\mathcal{D}_G(X)$), with $\varprojlim \mathcal{D}^+(\bar{E}_n|e_n)$ replaced by $\varprojlim \mathcal{D}^b(\bar{E}_n|e_n)$ (resp. $\varprojlim \mathcal{D}^-(\bar{E}_n|e_n)$; $\varprojlim \mathcal{D}(\bar{E}_n|e_n)$).

We call the objects in the categories just defined *equivariant complexes*.

The following results are proved in [S08] and [WW09].

Proposition 3.4. (a) *Acyclic sequences of smooth resolutions of X exist;*

(b) *The definitions of equivariant derived categories are independent of the choices of the acyclic sequences of smooth resolutions of X;*

(c) *The category $\mathcal{D}_G^+(X)$ (resp. $\mathcal{D}_G^b(X)$, $\mathcal{D}_G^-(X)$) is a triangulated category. Σ is a distinguished triangle in $\mathcal{D}_G^+(X)$ (resp. $\mathcal{D}_G^b(X)$, $\mathcal{D}_G^-(X)$) if and only if the projection of Σ to $\mathcal{D}^+(E_n|e_n)$ (resp. $\mathcal{D}^b(E_n|e_n)$, $\mathcal{D}^-(E_n|e_n)$) is a distinguished triangle for all $n \in \mathbb{N}$;*

For any object $(A_n, \phi_n) \in \mathcal{D}_G^+(X)$, there exists $B_n \in \mathcal{D}_G^+(X)$ such that $\bar{e}_n * A_n \simeq e_n^* B_n$ for any $n \in \mathbb{N}$. So $e_n^* B_n \simeq f_{n+1}^* \bar{e}_{n+1}^*(A_{n+1}) \simeq e_n^* B_{n+1}$. So the stalks of B_n and B_{n+1} at any

given points are isomorphic to each other. Hence $\mathrm{supp}(B_n) = \mathrm{supp}(B_{n+1})$, for all $n \in \mathbb{N}$. Therefore, we can define the support of the object (A_n, ϕ_n) as follows.

Definition 3.5. The *support* of the object $(A_n, \phi_n) \in \mathcal{D}^-_G(X)$ is defined to be the support of the B_n's.

Similarly, we can define the supports of any objects in $\mathcal{D}^b_G(X)$, $\mathcal{D}^+_G(X)$, and $\mathcal{D}_G(X)$.

Finally, we should give another description of the bounded derived category $\mathcal{D}^b_G(X)$ in order to talk about the singular supports of objects in $\mathcal{D}^b_G(X)$.

Let $X \overset{e}{\leftarrow} E \overset{\bar{e}}{\rightarrow} \bar{E}$ be a smooth resolution of X with respect to G. We define the category $\mathcal{D}^b_G(X, E)$ to be the category whose objects are

- triples $K = (K_X, \bar{K}, \theta)$, where $K_X \in \mathcal{D}^b(X)$, $\bar{K} \in \mathcal{D}^b(\bar{E})$ and $\theta : e^*K_X \to \bar{e}^*\bar{K}$ is an isomorphism;

and whose morphisms are

- pairs $\alpha = (\alpha_X, \bar{\alpha}) : K \to L$, with $\alpha_X : K_X \to L_X \in \mathcal{D}^b(X)$ and $\bar{\alpha} : \bar{K} \to \bar{L} \in \mathcal{D}^b(\bar{E})$ such that $\theta e^*(\alpha_X) = e^*(\bar{\alpha})\theta$.

Given any morphism $\nu : E \to E_1$ of smooth resolutions of X, it induces a morphism $\bar{\nu} : \bar{E} \to \bar{E}_1$, and a functor

$$\nu^* : \mathcal{D}^b_G(X, E_1) \to \mathcal{D}^b_G(X, E), \quad (K_X, \bar{K}, \theta) \to (K_X, \bar{\nu}^*\bar{K}, \xi),$$

where ξ is the composition $e^*(K_X) = \nu^* e_1^* K_X \overset{\nu^*\theta}{\to} \nu^* \bar{e}_1^*(\bar{K}) = e^* \bar{\nu}^* \bar{K}$.

An acyclic sequence (E_n, f_n) of smooth resolutions of X gives rise to a sequence of functors

$$\mathcal{D}^b_G(X, E_0) \xleftarrow{f_0^*} \mathcal{D}^b_G(X, E_1) \xleftarrow{f_1^*} \cdots \longleftarrow \mathcal{D}^b_G(X, E_n) \longleftarrow \cdots.$$

The following proposition is proved in [S08, 5.3] and [BL94, Part I, 2].

Proposition 3.6. *$\mathcal{D}^b_G(X) \simeq \varprojlim \mathcal{D}^b(X, E_n)$.*

We refer to [BL94, I, 5] for the definition of *equivariant perverse sheaves*. Recall that to any irreducible, G-invariant, closed subvariety X_1 in X, associated the *equivariant intersection cohomology*, denoted by $\mathrm{IC}_G(X_1)$. Let

$$\widetilde{\mathrm{IC}}_G(X_1) = \mathrm{IC}_G(X_1)[-\dim X_1]. \tag{2}$$

Note that if X is smooth, $\widetilde{\mathrm{IC}}_G(X) = \bar{\mathbb{Q}}_{l,X}$, the constant sheaf on X.

Remark 3.7. As in [LMB00, 18.7], the equivariant derived category $\mathcal{D}^*_G(X)$ for $* = \{\phi, +, -, b\}$ is equivalent to the derived category $D^*_c([G\backslash X], \bar{\mathbb{Q}}_l)$ of complexes of lisse-étale $\bar{\mathbb{Q}}_l$-sheaves on the quotient stack $[G\backslash X]$ of cartesian constructible cohomologies defined in [LMB00]. *We shall identify these two categories from now on.*

3.8. **General inverse and direct image functors.** Throughout this paper, we assume that for any homomorphism, say $\phi : H \to G$, of linear algebraic groups,

- $H = G \times G_1$ for some linear algebraic group G_1 and ϕ is the projection to G.

A morphism $f : Y \to X$ is called a *ϕ-map* if $f(h.y) = \phi(h).f(y)$ for any $h \in H$, $y \in Y$. Any ϕ-map $f : Y \to X$ gives rise to a morphism

$$Qf : [H\backslash Y] \to [G\backslash X] \tag{3}$$

of algebraic stacks ([LMB00]) defined as follows. For any given resolution $Y \xleftarrow{e} E$, the composition $X \xleftarrow{fe} E$ factors through the quotient map $q : E \to G_1\backslash E$ due to the assumption that G_1 acts trivially on X. In other words, there exists a unique morphism $X \xleftarrow{e_1} G_1\backslash E$ such that $e_1 q = fe$. Moreover, $H\backslash E = G\backslash(G_1\backslash E)$. For any morphism $\nu : E \to E'$ in $[H\backslash Y]$, it naturally induces a morphism $\bar{\nu} : \bar{E} \to \bar{E}'$ in $[G\backslash X]$. The morphism Qf is then defined to be

$$Qf : Y \xleftarrow{e} E \mapsto X \xleftarrow{e_1} G_1\backslash E, \quad \forall H\text{-resolution } Y \xleftarrow{e} E \in [H\backslash Y];$$
$$Qf : E \xrightarrow{\nu} E' \mapsto \bar{E} \xrightarrow{\bar{\nu}} \bar{E}', \quad \forall \text{ morphism } E \xrightarrow{\nu} E' \in [H\backslash Y].$$

By [LMB00], [LO08b], and Remark 3.7, the morphism Qf gives rise to the following functors:

$$\begin{aligned} &Qf_* : \mathcal{D}^+_H(Y) \to \mathcal{D}^+_G(X), \quad Qf_! : \mathcal{D}^-_H(Y) \to \mathcal{D}^-_G(X), \\ &Qf^* : \mathcal{D}_G(X) \to \mathcal{D}_H(Y), \quad Qf^! : \mathcal{D}_G(X) \to \mathcal{D}_H(Y). \end{aligned} \tag{4}$$

When $H = G$, we simply write f for the morphism Qf, and f_*, $f_!$, f^*, $f^!$ for the functors Qf_*, $Qf_!$, Qf^* and $Qf^!$, respectively. These functors satisfy the following properties. We refer to [LMB00] and [LO08b] for proofs of these properties.

Lemma 3.9. *The pairs (Qf^*, Qf_*) and $(Qf_!, Qf^!)$ are adjoint pairs.*

Lemma 3.10. $Qf^*(A \otimes A') = Qf^* A \otimes Qf^* A'$, *for any $A, A' \in \mathcal{D}^b_G(X)$.*

Lemma 3.11. *If, moreover, $\psi : I \to H$ is a morphism of linear algebraic groups, and $g : Z \to Y$ a ψ-map, we have*

$$Qg^* Qf^* \simeq Q(fg)^* \quad \textit{and} \quad Qf_* Qg_* \simeq Q(fg)_*.$$

Lemma 3.12. $Qf_!(A \otimes Qf^*(B)) \simeq Qf_!(A) \otimes B$, *for any $A \in \mathcal{D}^-_H(Y)$ and $B \in \mathcal{D}^-_G(X)$.*

Lemma 3.13. *If, moreover, $\psi : I \to H$ is a morphism of linear algebraic groups, and $g : Z \to Y$ a ψ-map, we have*

$$Qg^! Qf^! \simeq Q(fg)^! \quad \textit{and} \quad Qf_! Qg_! \simeq Q(fg)_!.$$

Suppose that

$$I = G \times G_1 \times G_2, \quad H = G \times G_1, \quad \text{and} \quad H_1 = G \times G_2.$$

We denote the projections of linear algebraic groups as follows.

$$I \xrightarrow{\psi} H \xrightarrow{\phi} G, \quad \text{and} \quad I \xrightarrow{\psi_1} H_1 \xrightarrow{\phi_1} G.$$

Assume, further, that we have a cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\ Y_1 & \xrightarrow{g_1} & X, \end{array}$$

and the morphisms f, g, f_1 and g_1 are ϕ, ψ, ψ_1 and ϕ_1 maps, respectively. This cartesian diagram then gives rise to the following cartesian diagram

$$\begin{array}{ccc} [I\backslash Z] & \xrightarrow{Qg} & [H\backslash Y] \\ {\scriptstyle Qf_1}\downarrow & & {\scriptstyle Qf}\downarrow \\ [H_1\backslash Y_1] & \xrightarrow{Qg_1} & [G\backslash X]. \end{array} \tag{5}$$

In fact, one can define a morphism,

$$\mathscr{X} \xrightarrow{\xi} [I\backslash Z],$$

from the fiber product $\mathscr{X} := [H_1\backslash Y_1] \times_{[G\backslash X]} [H\backslash Y]$ to $[I\backslash Z]$ as follows. To a triple $T = (E_1 \to Y_1, E \to Y, \alpha : Qg_1(E_1 \to Y_1) \simeq Qf(E \to Y))$ in the fiber product $\mathscr{X}$, we form the fiber product $E_1 \times_{(G_1\backslash E)} E$, on which there is a free I-action induced from the free H_1 (resp. H, G) action on E_1 (resp. E, $G_1\backslash E$). It is clear that $I\backslash E_1 \times_{(G_1\backslash E)} E \simeq H\backslash E$. By the universal property of the cartesian diagram $(g, f; f_1, g_1)$, we see that there is a unique I-equivariant morphism $E_1 \times_{(G_1\backslash E)} E \to Z$. The morphism ξ is defined by sending the triple T to $E_1 \times_{(G_1\backslash E)} E \to Z$. By the universal property of $\mathscr{X}$, there is a morphism $\xi_1 : [I\backslash Z] \to \mathscr{X}$. It is not difficult to show that $\xi\xi_1 = \mathrm{id}_{[I\backslash Z]}$ and $\xi_1\xi = \mathrm{id}_{\mathscr{X}}$. So $[I\backslash Z]$ is isomorphic to the fiber product $\mathscr{X}$. Therefore, the diagram above is cartesian.

The following lemma holds from the above cartesian diagram (5) of algebraic stacks.

Lemma 3.14. $Qf^*Qg_{1!} \simeq Qg_!Qf_1^*$.

3.15. **Compatibility with localization.** Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be three triangulated categories with functors

$$\mathcal{A} \xrightarrow{f_1} \mathcal{B} \xrightarrow{f_2} \mathcal{C}.$$

Suppose that $\mathcal{T}_{\mathcal{A}}$, $\mathcal{T}_{\mathcal{B}}$ and $\mathcal{T}_{\mathcal{C}}$ are thick subcategories of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, respectively. Then we have three "exact sequences" of categories ([BBD82, 1.4.4])

$$0 \to \mathcal{T}_{\mathcal{A}} \xrightarrow{\iota} \mathcal{A} \xrightarrow{Q} \mathcal{A}/\mathcal{T}_{\mathcal{A}} \to 0, \quad 0 \to \mathcal{T}_{\mathcal{B}} \xrightarrow{\iota} \mathcal{B} \xrightarrow{Q} \mathcal{B}/\mathcal{T}_{\mathcal{B}} \to 0, \quad 0 \to \mathcal{T}_{\mathcal{C}} \xrightarrow{\iota} \mathcal{C} \xrightarrow{Q} \mathcal{C}/\mathcal{T}_{\mathcal{C}} \to 0$$

Moreover, we assume that each localization functor Q admits a left adjoint $Q_!$ and a right adjoint Q_*. This implies that the functor ι admits a left adjoint ι^* and a right adjoint $\iota^!$ ([V76, §2]) . We form the following functors

$$F_1 = Q \circ f_1 \circ Q_! : \mathcal{A}/\mathcal{T}_{\mathcal{A}} \to \mathcal{B}/\mathcal{T}_{\mathcal{B}}; \quad F_2 = Q \circ f_2 \circ Q_! : \mathcal{B}/\mathcal{T}_{\mathcal{B}} \to \mathcal{C}/\mathcal{T}_{\mathcal{C}};$$
$$F_3 = Q \circ f_2 f_1 \circ Q_! : \mathcal{A}/\mathcal{T}_{\mathcal{A}} \to \mathcal{C}/\mathcal{T}_{\mathcal{C}}.$$

Lemma 3.16. *If $\iota^* f_1 Q_! = 0$ or $Q \circ f_2 \circ \iota = 0$, then $F_2 \circ F_1 = F_3$.*

Proof. From [V76, 6.7], we have a distinguished triangle of functors

$$Q_!Q \to \mathrm{Id} \to \iota\iota^* \to .$$

Thus we have a distinguished triangle of functors

$$Q \circ f_2(Q_!Q)f_1Q_! \to Q \circ f_2(\mathrm{Id})f_1Q_* \to Q \circ f_2(\iota\iota^*)f_1Q_! \to .$$

But the third term is zero by the assumption. Lemma follows. ∎

Suppose that $\mathcal{D}$ is a forth triangulated category with a thick subcategory $\mathcal{T}_{\mathcal{D}}$ such that the localization functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{T}_{\mathcal{D}}$ admits a right adjoint Q_* and a left adjoint $Q_!$. Assume, moreover, that we have a pair of functors $g_1 : \mathcal{A} \to \mathcal{D}, g_2 : \mathcal{D} \to \mathcal{C}$ such that $f_2 f_1 \simeq g_2 g_1$. We can form the following functors:

$$G_1 = Q \circ g_1 \circ Q_! \quad \text{and} \quad G_2 = Q \circ g_2 \circ Q_!.$$

Lemma 3.17. *If $Q \circ f_2 \circ \iota = 0$ and $\iota^* g_1 Q_! = 0$, then $F_2 F_1 \simeq G_2 G_1$.*

This follows from Lemma 3.16.

Assume that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\otimes} & \mathcal{A} \\ {\scriptstyle Q\otimes Q}\downarrow & & {\scriptstyle Q}\downarrow \\ \mathcal{A}/\mathcal{T}_{\mathcal{A}} \otimes \mathcal{A}/\mathcal{T}_{\mathcal{A}} & \xrightarrow{\otimes} & \mathcal{A}/\mathcal{T}_{\mathcal{A}}, \end{array} \qquad \begin{array}{ccc} \mathcal{B} \otimes \mathcal{B} & \xrightarrow{\otimes} & \mathcal{B} \\ {\scriptstyle Q\otimes Q}\downarrow & & {\scriptstyle Q}\downarrow \\ \mathcal{B}/\mathcal{T}_{\mathcal{B}} \otimes \mathcal{B}/\mathcal{T}_{\mathcal{B}} & \xrightarrow{\otimes} & \mathcal{B}/\mathcal{T}_{\mathcal{B}}. \end{array}$$

Lemma 3.18. *Suppose that $f_1(A \otimes A') = f_1 A \otimes f_1 A'$ and $Q \circ f_1 \circ \iota = 0$, then*

$$F_1(Q(A) \otimes Q(A')) \simeq F_1(A) \otimes F_1(A').$$

Proof. We have $F_1(Q(A) \otimes Q(A')) \simeq F_1 Q(A \otimes A') = Q \circ f_1 Q_! Q(A \otimes A')$. Consider the distinguished triangle

$$F_1(Q(A) \otimes Q(A') \to Q \circ f_1(A \otimes A') \to Q \circ f_1 \iota\iota^*(A \otimes A') \to .$$

By assumption, the third term of the above distinguished triangle is zero. So
$F_1(Q(A) \otimes Q(A')) \simeq Q \circ f_1(A \otimes A') = F(QA) \otimes F(QA')$. □

Lemma 3.19. *Assume that $Q_!(A \otimes Q(A') \simeq Q_!(A)) \otimes A'$ for any A, A' in $\mathcal{A}$. Suppose that $h_1 : \mathcal{B} \to \mathcal{A}$ is a functor. Let $H_1 = Q \circ h_1 \circ Q_!$. If $Q \circ h_1 \iota = 0$, $Q_! \circ F_1 = f_1 \circ Q_!$, and $f_1(A \otimes h_1(B)) \simeq f_1(A) \otimes B$, for $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then*

$$F_1(Q(A) \otimes H_1(Q(B))) \simeq F_1(Q(A)) \otimes Q(B).$$

Proof. By assumption, $H_1 \circ Q \simeq Q \circ h_1$. So

$$\begin{aligned} F_1(Q(A) \otimes H_1(Q(B))) &= F_1(Q(A) \otimes Q \circ h_1(B)) \simeq Q \circ f_1 \circ Q_!(Q(A) \otimes Q \circ h_1(B)) \\ &= Q \circ f_1(Q_! Q(A) \otimes h_1(B)) = Q \circ f_1(Q_! Q(A) \otimes h_1(B) \\ &= Q(f_1 Q_! Q(A) \otimes h_1(B) = Q(Q_! \circ F_1 Q(A) \otimes B) = F_1(Q(A)) \otimes Q(B). \end{aligned}$$

The lemma follows. □

4. Convolution product on localized equivariant derived categories

4.1. **Framed representation varieties.** Recall from section 2.1 that $\Gamma = (I, H, ', '', ^-)$ is a graph attached to a Cartan datum $(I, \cdot\,)$. Fix an orientation Ω of Γ, i.e., Ω is a subset of H such that $\Omega \sqcup \bar{\Omega} = H$. We call the pair (Γ, Ω) a *quiver*. To an I-graded vector space $V = \oplus_{i \in I} V_i$ over k, we set

$$G_V = \prod_{i \in I} \mathrm{GL}(V_i),$$

the product of the general linear groups $\mathrm{GL}(V_i)$. To a pair (D, V) of I-graded vector spaces over the field k, attached the *framed representation variety* of the quiver (Γ, Ω):

$$\mathbf{E}_\Omega(D, V) = \oplus_{h\in\Omega}\mathrm{Hom}(V_{h'}, V_{h''}) \oplus \oplus_{i\in I}\mathrm{Hom}(V_i, D_i).$$

Elements in $\mathbf{E}_\Omega(D, V)$ will be denoted by $X = (x, q)$ where x (resp. q) is in the first (resp. second) component. The group $G_D \times G_V$ acts on $\mathbf{E}_\Omega(D, V)$ by conjugation:

$$(f, g).(x, q) = (x', q'), \quad \text{where } x'_h = g_{h''}x_h g_{h'}^{-1},\ q'_i = f_i q_i g_i^{-1}, \quad \forall h \in \Omega, i \in I,$$

for any $(f, g) \in G_D \times G_V$, and $(x, q) \in \mathbf{E}_\Omega(D, V)$. To each $i \in I$, we set

$$X(i) = q_i + \sum_{h\in\Omega:h'=i} x_h : V_i \to D_i \oplus \bigoplus_{h\in\Omega:h'=i} V_{h''}.$$

To a triple (D, V, V') of I-graded vector spaces, we set

$$\mathbf{E}_\Omega(D, V, V') = \mathbf{E}_\Omega(D, V) \oplus \mathbf{E}_\Omega(D, V').$$

We write $\mathbf{E}_\Omega$ for $\mathbf{E}_\Omega(D, V, V')$ if it causes no ambiguity. The group

$$\mathbf{G} = G_D \times G_V \times G_{V'}$$

acts on $\mathbf{E}_\Omega$ by $(f, g, g').(X, X') = ((f, g).X, (f, g').X')$, for any $(f, g, g') \in \mathbf{G}$, $(X, X') \in \mathbf{E}_\Omega$.

Similarly, to a quadruple (D, V, V', V'') of I-graded vector spaces, we set

$$\mathbf{E}_\Omega(D, V, V', V'') = \mathbf{E}_\Omega(D, V) \oplus \mathbf{E}_\Omega(D, V') \oplus \mathbf{E}_\Omega(D, V'').$$

The group $\mathbf{H} = G_D \times G_V \times G_{V'} \times G_{V''}$ acts on $\mathbf{E}_\Omega(D, V, V', V'')$ by

$$(f, g, g', g'').(X, X', X'') = ((f, g).X, (f, g').X', (f, g'').X''),$$

for any $(f, g, g', g'') \in \mathbf{H}$, $(X, X', X'') \in \mathbf{E}_\Omega(D, V, V', V'')$.

4.2. **Fourier-Deligne transform.** Let Ω' be another orientation of the graph Γ. Let $\mathbf{E}_{\Omega\cup\Omega'}$, and $\mathbf{E}_{\Omega\cap\Omega'}$ be the spaces defined in a similar way as $\mathbf{E}_\Omega$. Thus we have the following diagram

$$\mathbf{E}_\Omega \xleftarrow{\pi} \mathbf{E}_{\Omega\cup\Omega'} \xrightarrow{\pi'} \mathbf{E}_{\Omega'}$$

where π (resp. π') is the projection map to the $\mathbf{E}_\Omega$ (resp. $\mathbf{E}_{\Omega'}$) component. It is clear that π and π' are $\mathbf{G}$-equivariant morphisms. Define a pairing

$$T : \mathbf{E}_{\Omega\cup\Omega'} \to k \tag{6}$$

by $T(X, X') = \sum_{h\in\Omega\setminus\Omega'} \mathrm{tr}(x'_h x'_{\bar{h}}) - \sum_{h\in\Omega\setminus\Omega'} \mathrm{tr}(x_h x_{\bar{h}})$ for any $(X, X') \in \mathbf{E}_{\Omega\cup\Omega'}$ where $\mathrm{tr}(-)$ is the trace of the endomorphism in the parenthesis. This pairing is $\mathbf{G}$-invariant, i.e., $T(g.(X, X')) = T(X, X')$, for any $g \in \mathbf{G}$. Via this pairing, we may regard $\mathbf{E}_{\Omega'}$ as the dual bundle of the vector bundle $\mathbf{E}_\Omega$ over $\mathbf{E}_{\Omega\cap\Omega'}$.

The Artin-Schreier covering $k \to k$ is defined by $x \mapsto x^p - x$. The Galois group of this covering is equal to the finite field $\mathbb{F}_p$ of p elements. Let $\chi : \mathbb{F}_p \to \bar{\mathbb{Q}}_l^*$ be a nontrivial additive character. It then induces a $\mathbf{G}$-equivariant local system, denoted by $\mathcal{L}_\chi$, on k. Let $\mathcal{L} = T^*(\mathcal{L}_\chi)$.

The *Fourier-Deligne transform* for the vector bundle $\mathbf{E}_\Omega$ over $\mathbf{E}_{\Omega\cap\Omega'}$, associated with the character χ, is the triangulated functor

$$\Phi_\Omega^{\Omega'} : \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega) \to \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_{\Omega'}) \tag{7}$$

defined by $\Phi_\Omega^{\Omega'}(K) = \pi'_!(\pi^*(K) \otimes \mathcal{L})[r]$, where r is the rank of the vector bundle $\mathbf{E}_\Omega \to \mathbf{E}_{\Omega\cap\Omega'}$.

Let a be the map of multiplication by -1 along the fiber of the vector bundle $\mathbf{E}_\Omega$ over $\mathbf{E}_{\Omega\cap\Omega'}$.

Theorem 4.3. *The transform $\Phi_\Omega^{\Omega'}$ is an equivalence of triangulated categories. Moreover, $\Phi_{\Omega'}^{\Omega}\Phi_\Omega^{\Omega'} \simeq a^*$.*

4.4. **Localization.** To each $i \in I$, we fix an orientation Ω_i of the graph Γ such that i is a *source*, i.e., all arrows h incident to i having $h' = i$. Let F_i be the closed subvariety of $\mathbf{E}_{\Omega_i}$ consisting of all elements (X, X') such that either $X(i)$ or $X'(i)$ is not injective. Let U_i be its complement. Thus we have a decomposition

$$\gamma_i : F_i \hookrightarrow \mathbf{E}_{\Omega_i} \hookleftarrow U_i \ : \beta_i. \tag{8}$$

Notice that F_i and U_i are $\mathbf{G}$-invariant. Following Zheng [ZH08], let $\mathcal{N}_i$ be the thick subcategory of $\mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)$ generated by the objects $K \in \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)$ such that the support of the complex $\Phi_\Omega^{\Omega_i}(K)$ is contained in the subvariety F_i. Let $\mathcal{N}$ be the thick subcategory of $\mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)$ generated by $\mathcal{N}_i$ for all $i \in I$. We define

$$\mathscr{D}^b \equiv \mathscr{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega) = \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)/\mathcal{N}$$

to be the localization of $\mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)$ with respect to the thick subcategory $\mathcal{N}$ ([V76], [KS90]). Let

$$Q : \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega) \to \mathscr{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)$$

denote the localization functor.

Lemma 4.5. *The localization functor Q admits a right adjoint Q_* and a left adjoint $Q_!$. Moreover, Q_* and $Q_!$ are fully faithful.*

Proof. Let $Q_i : \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega) \to \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)/\mathcal{N}_i$ be the localization functor with respect to the thick subcategory $\mathcal{N}_i$. It is well-known, for example, [BBD82, 1.4], [BL94, Thm. 3.4.3], that the functor $\beta_i^* : \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_{\Omega_i}) \to \mathcal{D}^b_{\mathbf{G}}(U_i)$ admits a fully faithful right adjoint β_{i*} and a fully faithful left adjoint $\beta_{i!}$. Now that the functor Q_i can be identified with the functor β_i^* via the transform $\Phi_\Omega^{\Omega_i}$, it then admits a fully faithful right adjoint Q_{i*} and a fully faithful left adjoint $Q_{i!}$.

For simplicity, let us assume that the graph Γ consists of only two vertices i and j. By the universal property of the localization functor Q_i, the functor Q factors through Q_i, i.e., there exists a functor $\hat{Q}_i : \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)/\mathcal{N}_i \to \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)/\mathcal{N}$ such that $Q = \hat{Q}_i \circ Q_i$. Similarly, we have $Q = \hat{Q}_j \circ Q_j$ for some $\hat{Q}_j : \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)/\mathcal{N}_j \to \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)/\mathcal{N}$. Moreover, the fact that Q_j admits a right adjoint Q_{j*} and a left adjoint implies that the functor $\hat{Q}_i$ admits a right adjoint $\hat{Q}_{i*}$ and a left adjoint $\hat{Q}_{i!}$.

In fact, let us consider the following diagram

$$\begin{tikzcd} \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega) \arrow[r, "Q_j"] & \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)/\mathcal{N}_j \arrow[r, "\hat{Q}_j"] \arrow[d, "Q_iQ_{j*}"'] & \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)/\mathcal{N} \\ & \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)/\mathcal{N}_i. & \end{tikzcd}$$

Given any $K \in \mathcal{N}$, it is clear that $Q_{j*}Q_j(K) \in \mathcal{N}$ and $\mathrm{supp}(Q_{j*}Q_j(K)) \cap U_j \neq \emptyset$. Thus we have $\mathrm{supp}(Q_{j*}Q_j(K)) \cap U_i = \emptyset$, i.e., $Q_iQ_{j*}Q_j(K) = 0$, for any $K \in \mathcal{N}$. From this, we

see that the functor $Q_i Q_{j*} Q_j$ factors through Q, i.e., there is a functor $\hat{Q}_{i*} : \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)/\mathcal{N} \to \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)/\mathcal{N}_i$ such that

$$\hat{Q}_{i*} Q = Q_i Q_{j*} Q_j. \tag{9}$$

By applying Q_{j*} to (9) and using the fact that $Q_j Q_{j*} \simeq \mathrm{id}$, we have $\hat{Q}_{i*}\hat{Q}_j = Q_i Q_{j*}$. One can show that $\hat{Q}_{i*}$ is the right adjoint of $\hat{Q}_i$. Moreover,

$$\hat{Q}_j \simeq \hat{Q}_j Q_j Q_{j*} \simeq \hat{Q}_i Q_i Q_{j*} \simeq \hat{Q}_i \hat{Q}_{i*} \hat{Q}_j.$$

This shows that $\mathrm{Id} \simeq \hat{Q}_i \hat{Q}_{i*}$, which is equivalent to say that Q_{i*} is fully faithful. The existence and the fully-faithfulness of the left adjoint $\hat{Q}_{i!}$ of $\hat{Q}_i$ can be proved similarly.

One can show that the functors $Q_* = Q_{i*} \circ \hat{Q}_{i*}$ and $Q_! = Q_{i!} \circ \hat{Q}_{i!}$ are the fully faithful right adjoint and left adjoint of Q, respectively.

In general, let us order the vertex set I as $i_1, \cdots, i_n$. Define a sequence of thick subcategories:

$$\mathcal{N}_1 \subseteq \cdots \subseteq \mathcal{N}_m \subseteq \cdots \subseteq \mathcal{N}_n = \mathcal{N}$$

where $\mathcal{N}_m$ is the thick subcategory generated by the subcategories $\mathcal{N}_{i_1}, \cdots, \mathcal{N}_{i_m}$. Then the functor Q is the composition of the following functors

$$\mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega) \to \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)/\mathcal{N}_1 \to \cdots \to \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)/\mathcal{N}_m \xrightarrow{Q_m} \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)/\mathcal{N}_{m+1} \to \cdots \to \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)/\mathcal{N}.$$

Each Q_m admits a fully faithful right adjoint Q_{m*} and a fully faithful left adjoint $Q_{m!}$ due to the fact that the functor $Q_{i_m} : \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega) \to \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)/\mathcal{N}_{i_m}$ admits fully faithful right and left adjoint functors. From this observation, we see that the functors

$$Q_* = Q_{1*} \circ \cdots \circ Q_{n*} \quad \text{and} \quad Q_! = Q_{1!} \circ \cdots \circ Q_{n!}$$

are the fully faithful right and left adjoint functors of Q, respectively. □

From the proof of Lemma 4.5, we see that

$$Q_*(K) = \prod_{i \in I} \Phi^{\Omega}_{\Omega_i} \beta_{i*} \beta_i^* \Phi^{\Omega_i}_{\Omega}(K) \quad \text{and} \quad Q_!(K) = \prod_{i \in I} \Phi^{\Omega}_{\Omega_i} \beta_{i!} \beta_i^* \Phi^{\Omega_i}_{\Omega}(K)$$

for any $K \in \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)$. Notice that the products are independent of the order of vertices in I. We also have

$$Q_!(A \otimes Q(B)) \simeq Q_!(A) \otimes B, \quad \forall A \in \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)/\mathcal{N}, B \in \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega). \tag{10}$$

This is because each pair $(Q_a, Q_{a!})$ in the proof of Lemma 4.5 has such a property. Moreover,

Lemma 4.6. (a) *The inclusion $\iota : \mathcal{N} \to \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)$ admits a left adjoint ι_* and a right adjoint $\iota^!$.*

(b) *One has $Q\iota = 0$, $\iota^* Q_! = 0$, $\iota^! Q_* = 0$, and, $\forall A \in \mathcal{N}$, $B \in \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)/\mathcal{N}$,*

$$\mathrm{Hom}(Q_! B, \iota A) = 0 \quad \textit{and} \quad \mathrm{Hom}(\iota A, Q_* B) = 0;$$

(c) *For any $K \in \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)$, there are distinguished triangles*

$$Q_! Q(K) \to K \to \iota\iota^*(K) \to \quad \textit{and} \quad \iota\iota^!(K) \to K \to Q_* Q(K) \to;$$

(d) *The functors $\iota, Q_*, Q_!$ are fully faithful, i.e., the following adjunction are isomorphic:*

$$\iota^*\iota \to Id \to \iota^!\iota \quad \textit{and} \quad QQ_* \to Id \to QQ_!.$$

Proof. These results follow from Lemma 4.5 and results from [V76, Prop. 6.5, 6.6, 6.7] and [KS06, Ch. 7, 10]. □

Let Ω' be another orientation. Let $\mathcal{N}_{\Omega'}$ be the thick subcategory defined in the same way as $\mathcal{N}$. One has, by definition, $\Phi_{\Omega}^{\Omega'}(\mathcal{N}) = \mathcal{N}_{\Omega'}$. So we have an equivalence, induced by $\Phi_{\Omega}^{\Omega'}$,

$$\Phi_{\Omega}^{\Omega'} : \mathscr{D}_{\mathbf{G}}^b(\mathbf{E}_{\Omega}) \simeq \mathscr{D}_{\mathbf{G}}^b(\mathbf{E}_{\Omega'}). \tag{11}$$

Similarly, one can define the category $\mathscr{D}_{\mathbf{H}}^b(\mathbf{E}_{\Omega}(D, V, V', V''))$ and the equivalence of categories $\Phi_{\Omega}^{\Omega'} : \mathscr{D}_{\mathbf{H}}^b(\mathbf{E}_{\Omega}(D, V, V', V'')) \simeq \mathscr{D}_{\mathbf{H}}^b(\mathbf{E}_{\Omega'}(D, V, V', V''))$.

Remark 4.7. (1). Comparing Lemma 4.5-4.6 with [BBD82, 1.4.3], [BL94], we may regard $\mathcal{D}_{\mathbf{G}}^b(\mathbf{E}_{\Omega})/\mathcal{N}$ as the equivariant derived category $\mathcal{D}_{\mathbf{G}}^b(U)$ of an "imaginary" open subvariety U of $\mathbf{E}_{\Omega}$.

(2). One may define $\mathscr{D}_{\mathbf{G}}^*(\mathbf{E}_{\Omega}(D, V, V'))$ for $* = \{\text{empty}, +, -\}$ in a similar way and Lemmas 4.5, 4.6 and (11) still hold.

4.8. **Convolution product.** Let

$$p_{ij} : \mathbf{E}_{\Omega}(D, V^1, V^2, V^3) \to \mathbf{E}_{\Omega}(D, V^i, V^j), \quad \phi : \mathbf{H} \to \mathbf{G},$$

denote the projections to the (i, j)-components, where the spaces $\mathbf{E}_{\Omega}$'s, the groups $\mathbf{H}$ and $\mathbf{G}$ are defined in Section 4.1. It is clear that p_{ij} is a ϕ-map. So, by Section 3.8 (3), we have a morphism of algebraic stacks:

$$Qp_{ij} : [\mathbf{H}\backslash \mathbf{E}_{\Omega}(D, V^1, V^2, V^3)] \to [\mathbf{G}\backslash \mathbf{E}_{\Omega}(D, V^i, V^j)].$$

From Section 3.8 (4), we have the following functors:

$$\begin{aligned} &Qp_{ij}^* : \mathcal{D}_{\mathbf{G}}^b(\mathbf{E}_{\Omega}(D, V^i, V^j)) \to \mathcal{D}_{\mathbf{H}}^b(\mathbf{E}_{\Omega}(D, V^1, V^2, V^3)); \\ &(Qp_{ij})_! : \mathcal{D}_{\mathbf{H}}^-(\mathbf{E}_{\Omega}(D, V^1, V^2, V^3)) \to \mathcal{D}_{\mathbf{G}}^-(\mathbf{E}_{\Omega}(D, V^i, V^j)). \end{aligned} \tag{12}$$

We set

$$\begin{aligned} &P_{ij}^* = Q \circ Qp_{ij}^* \circ Q_! : \mathscr{D}_{\mathbf{G}}^b(\mathbf{E}_{\Omega}(D, V^i, V^j)) \to \mathscr{D}_{\mathbf{H}}^b(\mathbf{E}_{\Omega}(D, V^1, V^2, V^3)); \\ &P_{ij!} = Q \circ (Qp_{ij})_! \circ Q_! : \mathscr{D}_{\mathbf{H}}^-(\mathbf{E}_{\Omega}(D, V^1, V^2, V^3)) \to \mathscr{D}_{\mathbf{G}}^-(\mathbf{E}_{\Omega}(D, V^i, V^j)). \end{aligned} \tag{13}$$

Lemma 4.9. *$P_{ij}^* \circ Q = Q \circ Qp_{ij}^*$, and $Q_! P_{ij!} = Qp_{ij!} Q_!$.*

Proof. We preserve the setting in the proof of Lemma 4.5. To each vertex i_a, we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{D}_{\mathbf{G}}^b(\mathbf{E}_{\Omega}(D, V^i, V^j)) & \xrightarrow{Qp_{ij}^*} & \mathcal{D}_{\mathbf{H}}^b(\mathbf{E}_{\Omega}(D, V^1, V^2, V^3)) \\ {\scriptstyle Q_{i_a}}\downarrow & & {\scriptstyle Q_{i_a}}\downarrow \\ \mathcal{D}_{\mathbf{G}}^b(\mathbf{E}_{\Omega}(D, V^i, V^j))/\mathcal{N}_{i_a} & \xrightarrow{f'_a} & \mathcal{D}_{\mathbf{H}}^b(\mathbf{E}_{\Omega}(D, V^1, V^2, V^3))/\mathcal{N}_{i_a}. \end{array}$$

The existence of f'_2 implies that we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{D}_{\mathbf{G}}^b(\mathbf{E}_{\Omega}(D, V^i, V^j))/\mathcal{N}_1 & \xrightarrow{f'_1} & \mathcal{D}_{\mathbf{H}}^b(\mathbf{E}_{\Omega}(D, V^1, V^2, V^3))/\mathcal{N}_1 \\ {\scriptstyle Q_2}\downarrow & & {\scriptstyle Q_2}\downarrow \\ \mathcal{D}_{\mathbf{G}}^b(\mathbf{E}_{\Omega}(D, V^i, V^j))/\mathcal{N}_2 & \xrightarrow{f_2} & \mathcal{D}_{\mathbf{H}}^b(\mathbf{E}_{\Omega}(D, V^1, V^2, V^3))/\mathcal{N}_2. \end{array}$$

Inductively, the existence of f'_a, for $a=2,\cdots,n$, gives rise to the following commutative diagram:

$$\begin{array}{ccc} \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega(D,V^i,V^j))/\mathcal{N}_{a-1} & \xrightarrow{f'_{a-1}} & \mathcal{D}^b_{\mathbf{H}}(\mathbf{E}_\Omega(D,V^1,V^2,V^3))/\mathcal{N}_1 \\ {\scriptstyle Q_a}\big\downarrow & & {\scriptstyle Q_a}\big\downarrow \\ \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega(D,V^i,V^j))/\mathcal{N}_a & \xrightarrow{f_a} & \mathcal{D}^b_{\mathbf{H}}(\mathbf{E}_\Omega(D,V^1,V^2,V^3))/\mathcal{N}_a. \end{array}$$

By composing these commutative diagrams, we have

$$Q\circ Qp^*_{ij}=f_nQ. \tag{14}$$

By applying the functor $Q_!$ to equation (14) and using Lemma 4.6 (d), we get

$$f_n\simeq f_nQQ_!=Q\circ Qp_{ij}\circ Q_!.$$

Therefore, $P^*_{ij}\circ Q=Q\circ Qp^*_{ij}$. The second identity can be proved in a similar way. □

From Lemma 4.9 and Lemma 4.6 (b), we have

$$Q\circ Qp^*_{ij}\circ\iota=0 \quad \text{and} \quad \iota^*\circ Qp_{ij!}\circ Q_!=0. \tag{15}$$

To any objects $K\in\mathscr{D}^-_{\mathbf{G}}(\mathbf{E}_\Omega(D,V^1,V^2))$ and $L\in\mathscr{D}^-_{\mathbf{G}}(\mathbf{E}_\Omega(D,V^2,V^3))$, associated

$$K\cdot L=P_{13!}(P^*_{12}(K)\otimes P^*_{23}(L)) \quad \in\mathscr{D}^-_{\mathbf{G}}(\mathbf{E}_\Omega(D,V^1,V^3)). \tag{16}$$

If, in addition, $M\in\mathscr{D}^-_{\mathbf{G}}(\mathbf{E}_\Omega(D,V^3,V^4))$, we have

Proposition 4.10. $(K\cdot L)\cdot M\simeq K\cdot(L\cdot M)$.

Proof. Let $X_i=\mathbf{E}_\Omega(D,V^i)$ and $G_i=G_{V^i}$, $\forall i=1,2,3,4$. Let

$$q_{ij}:X_1\times X_3\times X_4\to X_i\times X_j,\quad \phi_{ij}:G_D\times\prod_{i=1,3,4}G_i\to G_D\times\prod_{a=i,j}G_i;$$

$$r_{ijk}:X_1\times X_2\times X_3\times X_4\to X_i\times X_j\times X_k,\quad \phi_{ijk}:G_D\times\prod_{i=1,2,3,4}G_i\to G_D\times\prod_{a=i,j,k}G_a;$$

$$s_{ij}:X_1\times X_2\times X_3\times X_4\to X_i\times X_j,\quad \phi'_{ij}:G_D\times\prod_{i=1,2,3,4}G_i\to G_D\times G_i\times G_j;$$

be the self-explained projections. It is clear that q_{ij}, r_{ijk} and s_{ij} are ϕ_{ij}-map, ϕ_{ijk}-map, and ϕ'_{ij}-map, respectively. Similar to the functors $P_{ij!}$ and P^*_{ij}, we define the functors $Q_{ij!}$, Q^*_{ij} (resp. $R_{ijk!}$, R^*_{ijk}; $S_{ij!}$, S^*_{ij}) for the map q_{ij} (resp. r_{ijk}, s_{ij}). By definition,

$$(K\cdot L)\cdot M=Q_{14!}(Q^*_{13}(K\cdot L)\otimes Q^*_{34}(M))=Q_{14!}(Q^*_{13}P_{13!}(P^*_{12}(K)\otimes P^*_{23}(L))\otimes Q^*_{34}(M)).$$

Since the square $(r_{123},p_{13};r_{134},q_{13})$ is cartesian and by Lemma 3.14, we have

$$Qq^*_{13}Qp_{13!}\simeq Qr_{134!}Qr^*_{123}. \tag{17}$$

By (15), we have $Q\circ Qr^*_{123}\circ\iota=0$ and $\iota^*\circ Qp_{13!}\circ Q_!=0$. So by Lemma 3.17 and (17),

$$Q^*_{13}P_{13!}=R_{134!}R^*_{123}. \tag{18}$$

Thus

$$(K\cdot L)\cdot M\simeq Q_{14!}(R_{134!}R^*_{123}(P^*_{12}(K)\otimes P^*_{23}(L))\otimes Q^*_{34}(M)).$$

By Lemma 3.12, Lemma 3.19, and the fact that $Q\circ Qr^*_{134}\iota=0$,

$$(K\cdot L)\cdot M\simeq Q_{14!}R_{134!}(R^*_{123}(P^*_{12}(K)\otimes P^*_{23}(L))\otimes R^*_{134}Q^*_{34}(M)). \tag{19}$$

By Lemma 3.13, Lemma 3.16, and the fact that $\iota^* \circ Q r_{134!} Q_! = 0$, we have

$$Q_{14!} R_{134!} = S_{14!}. \tag{20}$$

By Lemma 3.10, Lemma 3.18 and the fact that $Q \circ Q r_{123}^* \iota = 0$, we have

$$R_{123}^*(P_{12}^* K \otimes P_{23}^* L) = R_{123}^* P_{12}^* K \otimes R_{123}^* P_{23}^* L. \tag{21}$$

By Lemma 3.11 and Lemma 3.16, we have

$$R_{123}^* P_{12}^* = S_{12}^*, \quad R_{123}^* P_{23}^* = S_{23}^*, \quad \text{and} \quad R_{134}^* Q_{34}^* = S_{34}^*. \tag{22}$$

Combining (19)-(22), we get

$$(K \cdot L) \cdot M \simeq S_{14!}(S_{12}^*(K) \otimes S_{23}^*(L) \otimes S_{34}^*(M)).$$

Similarly, we can show that $K \cdot (L \cdot M) \simeq S_{14!}(S_{12}^*(K) \otimes S_{23}^*(L) \otimes S_{34}^*(M))$. The lemma follows. $\square$

Let Ω' be another orientation of the graph Γ. We can define a similar convolution product, denoted by $\cdot_{\Omega'}$, on $\mathbf{E}_{\Omega'}$'s. We have

Proposition 4.11. $\Phi_\Omega^{\Omega'}(K \cdot L) = \Phi_\Omega^{\Omega'}(K) \cdot_{\Omega'} \Phi_\Omega^{\Omega'}(L)$, *for any objects* $K \in \mathscr{D}^-_{\mathbf{G}}(\mathbf{E}_\Omega(D, V^1, V^2))$ *and* $L \in \mathscr{D}^-_{\mathbf{G}}(\mathbf{E}_\Omega(D, V^2, V^3))$.

The proof of this proposition will be given in a forthcoming paper [Li10b]. It will not be used for the rest of the paper except Corollary 5.8.

4.12. **Conjectures.** Given any pair $(X, X') \in \mathbf{E}_\Omega(D, V, V')$, we write

$$\text{“ } X \hookrightarrow X' \text{ ”}$$

if there exists an inclusion $\rho : V \to V'$ such that $\rho_{h''} x_h = x'_h \rho_{h'}$, $q_i = q'_i \rho_i$, for any h in Ω and i in I. We also write $\rho : X \hookrightarrow X'$ for such a ρ. Consider the variety

$$\mathbf{Z}_\Omega \equiv \mathbf{Z}_\Omega(D, V, V') = \{(X, X', \rho) | (X, X') \in \mathbf{E}_\Omega(D, V, V') \text{ and } \rho : X \hookrightarrow X'\}. \tag{23}$$

Then we have a diagram

$$\mathbf{E}_\Omega(D, V) \xleftarrow{\pi_1} \mathbf{Z}_\Omega(D, V, V') \xrightarrow{\pi_{12}} \mathbf{E}_\Omega(D, V, V'),$$

where π_1 (resp. π_{12}) is the projection to the first (resp. $(1,2)$) component. Observe that π_1 is smooth with connected fibers. So $\mathbf{Z}_\Omega(D, V, V')$ is smooth and irreducible since $\mathbf{E}_\Omega(D, V)$ is smooth and irreducible.

Let $\mathbf{Z}^t_\Omega(D, V', V)$ be the variety obtained from $\mathbf{Z}_\Omega(D, V, V')$ by switching the first two components. Let π_{12} still denote the projection $\mathbf{Z}^t_\Omega(D, V', V) \to \mathbf{E}_\Omega(D, V', V)$.

Recall that Ω_i is an orientation such that i is a source.

Let λ be a dominant weight in $\mathbf{X}^+$ (2.1) such that $(\check{\alpha}_i, \lambda) = \dim D_i$ for all $i \in I$.

We set the following complexes in $\mathscr{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)$ with $\mu = \lambda - \nu$ and $n \in \mathbb{N}$:

$$\mathscr{I}^\Omega_\mu = Q\left(\pi_{12!}(\widetilde{\mathrm{IC}}_{\mathbf{G}}(\mathbf{Z}_\Omega))\right), \quad \text{if } \dim V = \dim V' = \nu;$$

$$\mathscr{E}^{(n),\Omega}_{\mu, \mu - n\alpha_i} = Q\left(\Phi^\Omega_{\Omega_i} \pi_{12!}(\widetilde{\mathrm{IC}}_{\mathbf{G}}(\mathbf{Z}_{\Omega_i}))[n(d_i + \sum_{h \in \Omega_i : h' = i} \nu_{h''} - (\nu_i + n))]\right),$$

if $\dim V=\nu$ and $\dim V'=\nu+ni$; and

$$\mathscr{F}^{(n),\Omega}_{\mu,\mu+n\alpha_i}=Q\left(\Phi^{\Omega}_{\Omega_i}\pi_{12!}(\widetilde{\mathrm{IC}}_{\mathbf{G}}(\mathbf{Z}^t_{\Omega_i}(D,V',V))[n(\nu_i-n)]\right),$$

if $\dim V=\nu-ni$ and $\dim V'=\nu$, where $\widetilde{\mathrm{IC}}_{\mathbf{G}}(-)$ is the shifted equivariant intersection cohomology of the variety inside the parenthesis defined in Section 3.2 (2). Note that $\widetilde{\mathrm{IC}}(\mathbf{Z}_\Omega)$ is nothing but the constant sheaf $\bar{\mathbb{Q}}_{l,\mathbf{Z}_\Omega}$ on the variety $\mathbf{Z}_\Omega$.

Consider the complexes of the form

$$K_1\cdot K_2\cdot\ldots\cdot K_m\quad\in\mathscr{D}^-_{\mathbf{G}}(\mathbf{E}_\Omega(D,V,V')),\tag{24}$$

where the K_a's are either $\mathscr{E}^{(n),\Omega}_{\bullet}$ or $\mathscr{F}^{(n),\Omega}_{\bullet}$.

Conjecture 4.13. *The complexes (24) are bounded and semisimple.*

We assume that Conjecture 4.13 holds for the rest of this section.

Let $d=\dim D$. Let $\mathscr{B}^{\Omega}_{d,\nu,\nu'}$ be the set of all isomorphism classes of simple perverse sheaves appeared as direct summands with possible shifts in the complexes (24) in $\mathscr{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega(D,V,V'))$ with $\nu=\dim V$ and $\nu'=\dim V'$. Let $\mathscr{K}^{\Omega}_{d,\nu,\nu'}$ be the free $\mathbb{A}$-module spanned by elements in $\mathscr{B}^{\Omega}_{d,\nu,\nu'}$. We set

$$\mathscr{B}^{\Omega}_d=\bigsqcup_{\nu,\nu'\in\mathbb{N}[I]}\mathscr{B}^{\Omega}_{d,\nu,\nu'}\quad\text{and}\quad\mathscr{K}^{\Omega}_d=\bigoplus_{\nu,\nu'\in\mathbb{N}[I]}\mathscr{K}^{\Omega}_{d,\nu,\nu'}.$$

The convolution product "$\cdot$" (16) induces a bilinear map

$$\cdot:\mathscr{K}^{\Omega}_{d,\nu,\nu'}\times\mathscr{K}^{\Omega}_{d,\nu',\nu''}\to\mathscr{K}^{\Omega}_{d,\nu,\nu''},\quad\forall\nu,\nu',\nu''\in\mathbb{N}[I],$$

by setting

$$[K_1]\cdot[K_2]=\sum_{n\in\mathbb{Z};[K]\in\mathscr{B}^{\Omega}_{d,\nu,\nu''}}G^{K,n}_{K_1,K_2}v^n[K],\quad\text{if } K_1\cdot K_2=\bigoplus_{n\in\mathbb{Z};[K]\in\mathscr{B}^{\Omega}_{d,\nu,\nu''}}K^{G^{K,n}_{K_1,K_2}}[n],$$

where $[K]$ denotes the isomorphism class of K and K^a is the direct sum of a copies of K. By adding up these bilinear maps, we have a bilinear map

$$\cdot:\mathscr{K}^{\Omega}_d\times\mathscr{K}^{\Omega}_d\to\mathscr{K}^{\Omega}_d.$$

By Proposition 4.10, the pair $(\mathscr{K}^{\Omega}_d,\cdot)$ is an associative algebra over $\mathbb{A}$. We make the following

Conjecture 4.14. *The assignments $1_\mu\mapsto\mathscr{I}^{\Omega}_\mu$, $E^{(n)}_{\mu,\mu-n\alpha_i}\mapsto\mathscr{E}^{(n),\Omega}_{\mu,\mu-n\alpha_i}$, $F^{(n)}_{\mu,\mu+n\alpha_i}\mapsto\mathscr{F}^{(n,\Omega)}_{\mu,\mu+n\alpha_i}$, and the rest of generators in ${}_{\mathbb{A}}\dot{\mathbf{U}}$ to 0, define a surjective $\mathbb{A}$-algebra homomorphism*

$$\Psi^{\Omega}_d:\ {}_{\mathbb{A}}\dot{\mathbf{U}}\to\mathscr{K}^{\Omega}_d;$$

where ${}_{\mathbb{A}}\dot{\mathbf{U}}$ is defined in section 2.2. Moreover, the image of $\dot{\mathbf{B}}$ under Ψ^{Ω}_d is $\mathscr{B}^{\Omega}_d$ with possible shifts, i.e., for any $b\in\dot{\mathbf{B}}$, $\Psi^{\Omega}_d(b)=v^{s(b)}[K]$ for some $s(b)\in\mathbb{Z}$ and $[K]\in\mathscr{B}^{\Omega}_d$.

Remark 4.15. (1). The definition of the generators $\mathscr{I}^{\Omega}_{\bullet}$, $\mathscr{E}^{(n),\Omega}_{\bullet}$ and $\mathscr{F}^{(n),\Omega}_{\bullet}$ is motivated by the work [ZH08]. In fact, define the thick subcategory $\mathcal{N}$ of $\mathcal{D}^b_{\mathbf{G}}(\mathbf{Z}_\Omega)$ by taking all objects K in $\mathcal{D}^b_{\mathbf{G}}(\mathbf{Z}_\Omega)$ such that $\pi_{12!}(K)\in\mathcal{N}\subset\mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)$. Let $Q:\mathcal{D}^b_{\mathbf{G}}(\mathbf{Z}_\Omega)\to\mathcal{D}^b_{\mathbf{G}}(\mathbf{Z}_\Omega)/\mathcal{N}$ be the localization functor. It is clear that the left adjoint $Q_!$ of Q exists. Then the generator $\mathfrak{F}^{(n)}_{\nu,i}:\mathfrak{D}_\nu\to\mathfrak{D}_{\nu+ni}$ in [ZH08] can be rewritten as

$$\mathfrak{F}^{(n)}_{\nu,i}(-)=P_{2!}(\mathscr{E}^{(n),\Omega}_{\mu,\mu-n\alpha_i}\otimes P_1^*(-))$$

where P_1^* and $P_{2!}$ are obtained from the projections of $\mathbf{E}_\Omega(D, V, V')$ to the first and second components, respectively, similar to the functors P_{ij}^* and $P_{ij!}$. There is a similar relation between the generator $\mathfrak{E}_{\nu,i}^{(n)}$ in [ZH08] and $\mathscr{F}_{\mu,\mu+n\alpha_i}^{(n),\Omega}$. From these relations and the results in [ZH08], it should be straightforward to show that the generators $\mathscr{I}_\bullet^\Omega$, $\mathscr{E}_\bullet^{(n),\Omega}$ and $\mathscr{F}_\bullet^{(n),\Omega}$ satisfy the defining relations of the corresponding quantum modified algebras. Details will be given in [Li10b].

(2) The proof of Lemma 5.6 shows that the generators $\mathscr{E}_\bullet^{(n),\Omega}$ and $\mathscr{F}_\bullet^{(n),\Omega}$ have the following simpler presentation:

$$\mathscr{E}_{\mu,\mu-n\alpha_i}^{(n),\Omega} = Q\left(\pi_{12!}(\bar{\mathbb{Q}}_{l,\mathbf{Z}_\Omega})[e_{\mu,ni}]\right) \quad \text{and} \quad \mathscr{F}_{\mu,\mu+n\alpha_i}^{(n),\Omega} = Q\left(\pi_{12!}(\bar{\mathbb{Q}}_{l,\mathbf{Z}_\Omega^t})[f_{\mu,ni}]\right),$$

where $e_{\mu,ni} = n(d_i + \sum_{h\in\Omega:h'=i}\nu_{h''} - (\nu_i+n))$ and $f_{\mu,ni} = n((\nu_i - n) - \sum_{h\in\Omega:h''=i}\nu_{h'})$.

(3). A consequence of Conjecture 4.14 is the positivity of the structure constants of the basis $\dot{\mathbf{B}}$.

5. Type A case

5.1. **Results from** [BLM90] **and** [SV00]**.** Let D be a d-dimensional vector space over k in this section. Let $\mathbf{S}_d$ be the set of all nondecreasing $N+1$ step sequences, $\underline{\nu} = (0 \leq \nu_1 \leq \cdots \leq \nu_N \leq d)$, of positive integers. To any $\underline{\nu} \in \mathbf{S}_d$, attached the partial flag variety $\mathcal{F}_{\underline{\nu}}$ consisting of all flags $F = (0 \equiv F_0 \subseteq F_1 \subseteq \cdots \subseteq F_N \subseteq F_{N+1} \equiv D)$ such that $\dim F_a = \nu_a$ for $a = 1, \cdots, N$. We set

$$\mathcal{F} = \sqcup_{\underline{\nu}\in\mathbf{S}_d}\mathcal{F}_{\underline{\nu}} \quad \text{and} \quad \mathcal{F}\times\mathcal{F} = \sqcup_{\underline{\nu},\underline{\nu}'\in\mathbf{S}_d}\mathcal{F}_{\underline{\nu}}\times\mathcal{F}_{\underline{\nu}'}$$

The general linear group $G_D = \mathrm{GL}(D)$ acts naturally on $\mathcal{F}$ from the left, and extends to a diagonal action on $\mathcal{F}\times\mathcal{F}$. Note that each piece $\mathcal{F}_{\underline{\nu}}\times\mathcal{F}_{\underline{\nu}'}$ in $\mathcal{F}\times\mathcal{F}$ is invariant under the G_D-action.

Let $q_{ij} : \mathcal{F}_{\underline{\nu}}\times\mathcal{F}_{\underline{\nu}'}\times\mathcal{F}_{\underline{\nu}''} \to \mathcal{F}_{\underline{\nu}^i}\times\mathcal{F}_{\underline{\nu}^j}$ be the projection to the (i,j)-components. We define a convolution product

$$\circ : \mathcal{D}^b_{G_D}(\mathcal{F}\times\mathcal{F}) \times \mathcal{D}^b_{G_D}(\mathcal{F}\times\mathcal{F}) \to \mathcal{D}^b_{G_D}(\mathcal{F}\times\mathcal{F}) \tag{25}$$

by $K\circ L = q_{13!}(q_{12}^*(K)\otimes q_{23}^*(L))$ for any $K, L \in \mathcal{D}^b_{G_D}(\mathcal{F}\times\mathcal{F})$. For $1\leq i\leq N$, $n\in\mathbb{N}$, we set

$$\begin{aligned}
\tilde{\Delta}_{\underline{\nu}} &= \{(F,F')\in\mathcal{F}_{\underline{\nu}}\times\mathcal{F}_{\underline{\nu}} | F = F'\},\\
\tilde{Y}_{\underline{\nu},-n\alpha_i} &= \{(F,F')\in\mathcal{F}_{\underline{\nu}}\times\mathcal{F}_{\underline{\nu'}} | F_j\subseteq F'_j, \dim F'_j/F_j = n\delta_{ij}, \forall 1\leq j\leq N\},\\
\tilde{Y}_{\underline{\nu},+n\alpha_i} &= \{(F,F')\in\mathcal{F}_{\underline{\nu}}\times\mathcal{F}_{\underline{\nu'}} | F_j\supseteq F'_j, \dim F_j/F'_j = n\delta_{ij}, \forall 1\leq j\leq N\},
\end{aligned}$$

if $\underline{\nu'}$ exists. Let

$$\begin{aligned}
1_{\underline{\nu}} &= \widetilde{\mathrm{IC}}_{G_D}(\tilde{\Delta}_{\underline{\nu}}) & &\in \mathcal{D}^b_{G_D}(\mathcal{F}_{\underline{\nu}}\times\mathcal{F}_{\underline{\nu}});\\
E^{(n)}_{\underline{\nu},-n\alpha_i} &= \widetilde{\mathrm{IC}}_{G_D}(\tilde{Y}_{\underline{\nu},-n\alpha_i})[n(\nu_{i+1}-(\nu_i+n))] & &\in \mathcal{D}^b_{G_D}(\mathcal{F}_{\underline{\nu}}\times\mathcal{F}_{\underline{\nu'}});\\
F^{(n)}_{\underline{\nu},+n\alpha_i} &= \widetilde{\mathrm{IC}}_{G_D}(\tilde{Y}_{\underline{\nu},+n\alpha_i})[n((\nu_i-n)-\nu_{i-1})] & &\in \mathcal{D}^b_{G_D}(\mathcal{F}_{\underline{\nu}}\times\mathcal{F}_{\underline{\nu'}}),
\end{aligned}$$

where $\nu_{N+1} = d$ and $\nu_0 = 0$.

Let $\mathcal{Q}_{d,\underline{\nu},\underline{\nu}'}$ be the full subcategory of $\mathcal{D}^b_{G_D}(\mathcal{F}_{\underline{\nu}} \times \mathcal{F}_{\underline{\nu}'})$ consists of semisimple complexes whose simple components, up to a shift, appearing in the semisimple complexes of the form

$$K_1 \circ K_2 \circ \cdots \circ K_m \quad \in \mathcal{D}^b_{G_D}(\mathcal{F}_{\underline{\nu}} \times \mathcal{F}_{\underline{\nu}'}),$$

where K_a's runs over the various $E^{(n)}_{\underline{\nu},-n\alpha_i}$'s and $F^{(n)}_{\underline{\nu},+n\alpha_i}$'s and the category $\mathcal{D}^b_{G_D}(\mathcal{F}_{\underline{\nu}} \times \mathcal{F}_{\underline{\nu}'})$ is identified with the full subcategory of $\mathcal{D}^b_{G_D}(\mathcal{F} \times \mathcal{F})$ consisting of all complexes whose supports are contained in $\mathcal{F}_{\underline{\nu}} \times \mathcal{F}_{\underline{\nu}'}$.

Let $\mathcal{B}_{d,\underline{\nu},\underline{\nu}'}$ be the set of all isomorphism classes of simple perverse sheaves in $\mathcal{Q}_{d,\underline{\nu},\underline{\nu}'}$. Let $\mathcal{K}_{d,\underline{\nu},\underline{\nu}'}$ be the free $\mathbb{A}$-module spanned by elements in $\mathcal{B}_{d,\underline{\nu},\underline{\nu}'}$. We set

$$\mathcal{B}_d = \bigsqcup_{\underline{\nu},\underline{\nu}' \in \mathbf{S}_d} \mathcal{B}_{d,\underline{\nu},\underline{\nu}'} \quad \text{and} \quad \mathcal{K}_d = \bigoplus_{\underline{\nu},\underline{\nu}' \in \mathbf{S}_d} \mathcal{K}_{d,\underline{\nu},\underline{\nu}'}.$$

By a similar argument as in Section 4.12, we have an $\mathbb{A}$-bilinear map

$$\circ : \mathcal{K}_d \times \mathcal{K}_d \to \mathcal{K}_d.$$

Let $\dot{\mathbf{U}}(\mathfrak{sl}_{N+1})$ be the algebra $\dot{\mathbf{U}}$ defined in section 2.2 with respect to the graph Γ of type $\mathbf{A}_N$, i.e., $1 \leftrightarrows 2 \leftrightarrows \cdots \leftrightarrows N$. Let $I = \{1, 2, \cdots, N\}$ be the vertex of the graph Γ. Recall that $\lambda \in \mathbf{X}^+$ is the element such that $\langle \check{\alpha}_i, \lambda \rangle = 0$ for $i < N$ and $\langle \check{\alpha}_N, \lambda \rangle = d$.

Theorem 5.2 ([BLM90], [SV00]). *The pair $(\mathcal{K}_d, \circ)$ is an associative algebra over $\mathbb{A}$ and the assignments*

$$1_{\lambda-\nu} \mapsto 1_{\underline{\nu}}, \quad E^{(n)}_{\lambda-\nu,\lambda-\nu-n\alpha_i} \mapsto E^{(n)}_{\underline{\nu},-n\alpha_i}, \quad F^{(n)}_{\lambda-\nu,\lambda-\nu+n\alpha_i} \mapsto F^{(n)}_{\underline{\nu},+n\alpha_i}, \quad \forall \nu \in \mathbf{S}_d,$$

and all other generators in $\dot{\mathbf{U}}(\mathfrak{sl}_{N+1})$ send to 0, define a surjective $\mathbb{A}$-algebra homomorphism

$$\Psi_d : {}_{\mathbb{A}}\dot{\mathbf{U}}(\mathfrak{sl}_{N+1}) \to \mathcal{K}_d.$$

Moreover, the image of $\dot{\mathbf{B}}$ under Ψ_d is 0 or elements in $\mathcal{B}_d$ with possible shifts, i.e., for any $b \in \dot{\mathbf{B}}$, $\Psi_d(b) = 0$ or $v^{s(b)}[K]$ for some $s(b) \in \mathbb{Z}$ and $[K] \in \mathcal{B}_d$.

Note that the algebra $(\mathcal{K}_d, \circ)$ is the $\mathbb{A}$-form of the q-Schur algebra defined in [DJ89].

5.3. **Analysis of the pair $(\mathcal{K}^\Omega_d, \mathcal{B}^\Omega_d)$.** In this subsection, we assume that

- the graph Γ is of type $\mathbf{A}_N$ and D, a d-dimensional vector space over k.

We fix an orientation Ω of Γ as follows: $1 \to 2 \to \cdots \to N$. Let

$$\Omega^2 : 1 \to 2 \to \cdots \to N \to (N+1) \leftarrow N' \leftarrow \cdots \leftarrow 2' \leftarrow 1'. \tag{26}$$

To a pair (V, V') of I-graded vector spaces, the space $\mathbf{E}_\Omega(D, V, V')$ defined in section 4.1 is the representation variety of Ω^2 with V_a attached to the vertex a and V'_a to the vertex a' and D to the vertex $N+1$. (In the language of Section 4.1, we regard D as an I-graded vector space concentrated at the vertex N.)

Let us fix some notations. We will use $x_{a+1,a}$ to denote elements in $\mathrm{Hom}(V_a, V_{a+1})$. In particular, the element $q_N \in \mathrm{Hom}(V_N, D_N)$ is denoted by $x_{N+1,N}$ in this section. For a pair (i, j) such that $i \leq j$, $1 \leq i \leq N$ and $1 \leq j \leq N+1$, we fix a representative $S_{i,j}$ for the indecomposable representation of Ω^2 of dimension equal to 1 at the vertices $i, (i+1), \cdots, j$, and equal to 0 otherwise. The notation $S_{i',j'}$ is defined similarly. For $1 \leq i, j \leq N+1$, let $T_{i,j'}$ denote the indecomposable representation of the quiver Ω^2 such that the dimension of

$T_{i,j'}$ equals 1 at the vertices $i, i+1, \cdots, N, (N+1), N', (N-1)', \cdots j'$, and zero otherwise. When $i = j$, we simply write S_i for $S_{i,i}$, and T_i for $T_{i,i'}$.

Let

$$U = \{(X, X') \in \mathbf{E}_\Omega(D, V, V') | x_{a+1,a}, x'_{a+1,a}, \text{ are injective}, \forall a = 1, \cdots, N\}.$$

It is clear that U is nonempty only when $\dim V_1 \leq \dim V_2 \leq \cdots \dim V_N \leq \dim D_N = d$ and the same property for V'. The set of isomorphism classes, $[V]$, of I-graded vector spaces V of such a property is then in bijection with the set $\mathbf{S}_d$ defined in the section 5.1, via the map

$$[V] \mapsto |V| = (\dim V_1, \dim V_2, \cdots, \dim V_N).$$

By abuse of notations, we write $V \in \mathbf{S}_d$ if $\dim V_i \leq \dim V_j$ for $i \leq j$.

For any pair $V, V' \in \mathbf{S}_d$ such that $|V| = \underline{\nu}$ and $|V'| = \underline{\nu}'$, define a morphism of varieties

$$u : U \to \mathcal{F}_{\underline{\nu}} \times \mathcal{F}_{\underline{\nu}'}; \quad (X, X') \mapsto (F, F'),$$

where $F = (0 \subseteq \operatorname{im}(x_{N+1,N} x_{N,N-1} \cdots x_{2,1}) \subseteq \cdots \subseteq \operatorname{im}(x_{N+1,N}) \subseteq D_N)$, and F' is defined similarly. It is well-known ([N94]) that the algebraic group $\mathbf{G}_1 = G_V \times G_{V'}$ acts freely on U, and u can be identified with the quotient map $q : U \to \mathbf{G}_1 \backslash U$.

The following diagram of morphisms

$$\mathbf{E}_\Omega(D, V, V') \xleftarrow{\beta} U \xrightarrow{u} \mathcal{F}_{\underline{\nu}} \times \mathcal{F}_{\underline{\nu}'}$$

induces a diagram of morphisms of algebraic stacks

$$[\mathbf{G} \backslash \mathbf{E}_\Omega(D, V, V)] \xleftarrow{\beta} [\mathbf{G} \backslash U] \xrightarrow{Qu} [G_D \backslash \mathcal{F}_{\underline{\nu}} \times \mathcal{F}_{\underline{\nu}'}],$$

which, in turn, gives rise to the following diagram of functors

$$\mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega(D, V, V')) \xrightarrow{\beta^*} \mathcal{D}^b_{\mathbf{G}}(U) \xleftarrow{Qu^*} \mathcal{D}^b_{G_D}(\mathcal{F}_{\underline{\nu}} \times \mathcal{F}_{\underline{\nu}'}).$$

Lemma 5.4. *Suppose that K is a $\mathbf{G}$-equivariant complex on $\mathbf{E}_\Omega(D, V, V')$. Then $K \in \mathcal{N}$ in Section 4.4 if and only if K satisfies that* $\operatorname{supp}(K) \cap U = \emptyset$.

Proof. To each a, we fix an orientation of the graph Γ:

$$\Omega_a : 1 \to 2 \to \cdots \to (a-1) \leftarrow a \to (a+1) \to \cdots \to N.$$

Let W_a^Ω (resp. $W_a^{\Omega \cup \Omega_a}$) be the open subvariety of $\mathbf{E}_\Omega$ (resp. $\mathbf{E}_{\Omega \cup \Omega_a}$) consisting of all elements such that the component $x_{a+1,a}$ is injective. We have the following diagram

$$\begin{array}{ccccc}
W_a^\Omega & \xleftarrow{\tau} & W_a^{\Omega \cup \Omega_a} & \xrightarrow{\tau'} & W_a^{\Omega_a} \\
{\scriptstyle\beta}\big\downarrow & & \big\downarrow & & {\scriptstyle\beta}\big\downarrow \\
\mathbf{E}_\Omega & \xleftarrow{\pi} & \mathbf{E}_{\Omega \cup \Omega_a} & \xrightarrow{\pi'} & \mathbf{E}_{\Omega_a},
\end{array}$$

where the β's are the open inclusions, and τ (resp. τ') is the restriction of π (resp. π') to the variety $W_a^{\Omega \cup \Omega_a}$. Moreover, the squares in the above diagram are cartesian. From this diagram, we have

$$\beta^* \Phi_\Omega^{\Omega_a}(K) = \tau'_!(\tau^* \beta^* K \otimes \beta^* \mathcal{L})[r].$$

This implies that

$$\operatorname{supp}(\Phi_\Omega^{\Omega_a}(K)) \cap W_a^{\Omega_a} = \emptyset \quad \text{if and only if} \quad \operatorname{supp}(K) \cap W_a^\Omega = \emptyset. \tag{27}$$

Assume that $K \in \mathcal{N}_a$, then by (27), $\mathrm{supp}(K) \cap W_a^\Omega = \emptyset$. Since $W_a^\Omega \supseteq U$, we have $\mathrm{supp}(K) \cap U = \emptyset$. Since $\mathcal{N}$ is generated by the $\mathcal{N}_a$'s, we see that any object $K \in \mathcal{N}$ has the property that $\mathrm{supp}(K) \cap U = \emptyset$.

Now we shall show that if $\mathrm{supp}(K) \cap U = \emptyset$, then $K \in \mathcal{N}$. Since any object in $\mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega)$ can be generated by the $\mathbf{G}$-equivariant simple perverse sheaves, it is enough to show this statement for K a simple perverse sheaf, which we shall assume from now on.

It is well known that the $\mathbf{G}$-equivariant simple perverse sheaves on $\mathbf{E}_\Omega(D, V, V')$ are the intersection cohomology complexes $\mathrm{IC}_{\mathbf{G}}(\overline{\mathcal{O}_{(X,X')}})$, attached to the $\mathbf{G}$-orbit $\mathcal{O}_{(X,X')}$ in $\mathbf{E}_\Omega(D, V, V')$ containing the element (X, X'). (This is due to the facts that there is only finitely many $\mathbf{G}$-orbits in $\mathbf{E}_\Omega(D, V, V')$ and that the stabilizers of the orbits $\mathcal{O}$ in $\mathbf{E}_\Omega(D, V, V')$ are connected.) Therefore, $K \in \mathcal{N}$ if the following claim holds:

Claim. If $X(j) + X'(j')$ are injective for $1 \leq j \leq i-1$, and $X(i) + X'(i')$ is not injective for some $i \in [1, N]$, then $\mathrm{IC}_{\mathbf{G}}(\overline{\mathcal{O}_{(X,X')}}) \in \mathcal{N}_i$.

The claim can be shown as follows. Let $\mathbf{i} = (i_1, \cdots, i_n)$ be a sequence of vertices in Ω^2 and $\mathbf{a} = (a_1, \cdots, a_n)$ be a sequence of positive integers such that $\sum_{i_l = i} a_l = \dim V_i$ for any vertex i. Let $\widetilde{\mathcal{F}}^\Omega_{\mathbf{i},\mathbf{a}}$ be the variety of all triples $(X, X', \mathbf{F})$, where $(X, X') \in \mathbf{E}_\Omega(D, V, V')$ and $\mathbf{F} = (\mathbf{F}^0 = D \oplus V \oplus V' \supset \mathbf{F}^1 \supset \cdots \supset \mathbf{F}^n = 0)$ is a flag of graded vector subspaces in $D \oplus V \oplus V'$, such that

$$\dim \mathbf{F}^{l-1}/\mathbf{F}^l = a_l i_l \quad \text{and} \quad (X, X')(\mathbf{F}^l) \subset \mathbf{F}^l, \qquad \forall l = 1, \cdots, n.$$

Consider the projection to the $(1, 2)$-components:

$$\pi^\Omega_{\mathbf{i},\mathbf{a}} : \widetilde{\mathcal{F}}^\Omega_{\mathbf{i},\mathbf{a}} \to \mathbf{E}_\Omega(D, V, V'), \quad (X, X', \mathbf{F}) \mapsto (X, X'). \tag{28}$$

By [R03, Theorem 2.2], one can choose a particular pair $(\mathbf{i}, \mathbf{a})$ such that the image of $\pi^\Omega_{\mathbf{i},\mathbf{a}}$ is the closure $\overline{\mathcal{O}_{(X,X')}}$ of the orbit $\mathcal{O}_{(X,X')}$, and the restriction $(\pi^\Omega_{\mathbf{i},\mathbf{a}})^{-1}(\overline{\mathcal{O}_{(X,X')}}) \to \overline{\mathcal{O}_{(X,X')}}$ is an isomorphism. Thus, the complex $\mathrm{IC}_{\mathbf{G}}(\overline{\mathcal{O}_{(X,X')}})$ is a direct summand of the semisimple complex $(\pi^\Omega_{\mathbf{i},\mathbf{a}})_!(\bar{\mathbb{Q}}_{l,\widetilde{\mathcal{F}}^\Omega_{\mathbf{i},\mathbf{a}}})$, up to a shift.

So the complex $\Phi^{\Omega_i}_\Omega(\mathrm{IC}_{\mathbf{G}}(\overline{\mathcal{O}_{(X,X')}}))$ is a direct summand of the complex $\Phi^{\Omega_i}_\Omega((\pi^\Omega_{\mathbf{i},\mathbf{a}})_!(\bar{\mathbb{Q}}_{l,\widetilde{\mathcal{F}}^\Omega_{\mathbf{i},\mathbf{a}}}))$, up to a shift. By [L93, 10.2], the complex $\Phi^{\Omega_i}_\Omega((\pi^\Omega_{\mathbf{i},\mathbf{a}})_!(\bar{\mathbb{Q}}_{l,\widetilde{\mathcal{F}}^\Omega_{\mathbf{i},\mathbf{a}}}))$ is isomorphic to the complex $(\pi^{\Omega_i}_{\mathbf{i},\mathbf{a}})_!(\bar{\mathbb{Q}}_{l,\widetilde{\mathcal{F}}^{\Omega_i}_{\mathbf{i},\mathbf{a}}})$, up to a shift, where $\pi^{\Omega_i}_{\mathbf{i},\mathbf{a}}$ and $\widetilde{\mathcal{F}}^{\Omega_i}_{\mathbf{i},\mathbf{a}}$ are defined in exactly the same manner as $\pi^\Omega_{\mathbf{i},\mathbf{a}}$ and $\widetilde{\mathcal{F}}^\Omega_{\mathbf{i},\mathbf{a}}$ with $\mathbf{E}_\Omega(D, V, V')$ replaced by $\mathbf{E}_{\Omega_i}(D, V, V')$. So the support of the complex $\Phi^{\Omega_i}_\Omega(\mathrm{IC}_{\mathbf{G}}(\overline{\mathcal{O}_{(X,X')}}))$ is contained in the image of the morphism $\pi^{\Omega_i}_{\mathbf{i},\mathbf{a}}$.

Observe that the conditions in the claim imply that

- either $S_{1,i}$ or $S_{1',i'}$ is a direct summand of the representation M of Ω^2 corresponding to the element (X, X') in the claim;
- the representation M does not contain any direct summand of the form $S_{1,t}$ and $S_{1',t'}$ for $1 \leq t < i$.

From this observation, we see that the chosen sequence $\mathbf{i} = (i_1, \cdots, i_n)$ satisfies that $i_n = i$ or i', from the construction in [R03] and the Auslander-Reiten quiver of Ω^2. This implies that for any $(X, X', \mathbf{F}) \in \widetilde{\mathcal{F}}^{\Omega_i}_{\mathbf{i},\mathbf{a}}$, either $\ker(X(i)) \neq 0$ if $i_n = i$ or $\ker(X'(i')) \neq 0$ if $i_n = i'$. Therefore, the image of $\pi^{\Omega_i}_{\mathbf{i},\mathbf{a}}$ is contained in the subvariety of $\mathbf{E}_{\Omega_i}(D, V, V')$ consisting of all elements (X, X') such that $\ker(X(i)) \neq 0$ or $\ker(X'(i')) \neq 0$. The claim follows. ∎

By Lemma 5.4, there is an equivalence of triangulated categories

$$\bar{Q} : \mathcal{D}^b_{\mathbf{G}}(U) \to \mathscr{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega(D, V, V')) \equiv \mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega(D, V, V'))/\mathcal{N}$$

such that $Q = \bar{Q}\beta^*$ and $\beta_! = Q_!\bar{Q}$. Thus, we have a sequence of functors of equivalence

$$\mathcal{D}^b_{G_D}(\mathcal{F}_{\underline{\nu}} \times \mathcal{F}_{\underline{\nu}'}) \xrightarrow{Qu^*} \mathcal{D}^b_{\mathbf{G}}(U) \xrightarrow{\bar{Q}} \mathscr{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega(D, V, V')). \tag{29}$$

Define a convolution product "$\cdot$" on the categories $\mathcal{D}^b_{\mathbf{G}}(U_i)$ as follows. We have the following commutative diagram

$$\begin{tikzcd} U \arrow[r, "u_{ij}"] \arrow[d, "\beta"'] & U \arrow[d, "\beta"'] \\ \mathbf{E}_\Omega(D, V, V', V'') \arrow[r, "p_{ij}"] & \mathbf{E}_\Omega(D, V^i, V^j), \end{tikzcd}$$

where p_{ij} is the projection to the (i, j)-component and u_{ij} is the restriction of p_{ij} to U. The morphism u_{ij} then defines a morphism $Qu_{ij} : [\mathbf{H}\backslash U] \to [\mathbf{G}\backslash U]$. To any objects K and L in $\mathcal{D}^b_{\mathbf{G}}(U)$, associated an object

$$K \cdot L = Qu_{13!}(Qu^*_{12}(K) \otimes Qu^*_{23}(L)) \quad \in \mathcal{D}^b_{\mathbf{G}}(U).$$

Lemma 5.5. *The convolution products on $\mathcal{D}^b_{G_D}(\mathcal{F} \times \mathcal{F})$, $\mathcal{D}^b_{\mathbf{G}}(U)$ and $\mathscr{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega(D, V, V'))$ are compatible with the functors Qu^* and $\bar{Q}$ in the diagram (29).*

Proof. First, we show that the convolution products are compatible with the functors $\bar{Q}$. Recall that $Q = \bar{Q} \circ \beta^*$ and $\beta_! = Q_!\bar{Q}$. We have

$$\bar{Q} \circ Qu^*_{ij} = \bar{Q} \circ Qu^*_{ij}\beta^*\beta_! = \bar{Q}\beta^* \circ Qp^*_{ij}\beta_! = Q \circ Qp^*_{ij} \circ Q_!\bar{Q} = P^*_{ij}\bar{Q}. \tag{30}$$

Similarly,

$$\bar{Q} \circ Qu_{ij!} = \bar{Q}\beta^*\beta_! Qu_{ij!} = Q \circ Qp_{ij!}\beta_! = Q \circ Qp_{ij!}Q_!\bar{Q} = P_{ij!}\bar{Q}. \tag{31}$$

From (30) and (31), we see that the convolution products on $\mathcal{D}^b_{\mathbf{G}}(U_i)$ and $\mathscr{D}^b_{\mathbf{G}}(\mathbf{E}_\Omega(D, V, V'))$ are compatible with the functor $\bar{Q}$.

Now, we show that the convolution products are compatible with the functors Qu^*. We have the following cartesian diagram:

$$\begin{tikzcd} {[\mathbf{H}\backslash U]} \arrow[r, "Qu_{ij}"] \arrow[d, "Qu"'] & {[\mathbf{G}\backslash U]} \arrow[d, "Qu"'] \\ {[G_D\backslash(\mathcal{F}_{\underline{\nu}} \times \mathcal{F}_{\underline{\nu}'} \times \mathcal{F}_{\underline{\nu}''})]} \arrow[r, "q_{ij}"] & {[G_D\backslash(\mathcal{F}_{\underline{\nu}^i} \times \mathcal{F}_{\underline{\nu}^j})]}. \end{tikzcd}$$

where the first U is contained in $\mathbf{E}_\Omega(D, V, V', V'')$ such that $|V| = \underline{\nu}$, $|V'| = \underline{\nu}'$ and $|V''| = \underline{\nu}''$. This cartesian diagram gives rise to the following identities:

$$Qu^*q^*_{ij} = Qu^*_{ij}Qu^*, \quad \text{and} \quad Qu^*q_{ij!} = Qu_{ij!}Qu^*.$$

So for any $K \in \mathcal{D}^b_{G_D}(\mathcal{F}_{\underline{\nu}} \times \mathcal{F}_{\underline{\nu}'})$ and $L \in \mathcal{D}^b_{G_D}(\mathcal{F}_{\underline{\nu}'} \times \mathcal{F}_{\underline{\nu}''})$, we have

$$\begin{aligned} Qu^*(K \circ L) &= Qu^*q_{13!}(q^*_{12}(K) \otimes q^*_{13}(L)) = Qu_{13!}Qu^*(q^*_{12}(K) \otimes q^*_{13}(L)) \\ &= Qu_{13!}(Qu^*_{12}Qu^*(K) \otimes Qu^*_{23}Qu^*(L)) = Qu^*(K) \cdot Qu^*(L). \end{aligned}$$

Therefore, the convolution products commute with the functor Qu^*. The lemma follows. $\square$

We set $\mu = \lambda - \nu$, and

$$\mathscr{I}_\mu = \bar{Q} Q u^*(1_{\underline{\nu}}), \quad \mathscr{E}^{(n)}_{\mu,\mu-n\alpha_i} = \bar{Q} Q u^* \left(E^{(n)}_{\underline{\nu},-n\alpha_i} \right), \quad \text{and} \quad \mathscr{F}^{(n)}_{\mu,\mu+n\alpha_i} = \bar{Q} Q u^* \left(F^{(n)}_{\underline{\nu},+n\alpha_i} \right). \tag{32}$$

where λ, ν, $1_{\underline{\nu}}$, $E_\bullet$ and $F_\bullet$'s are from the section 5.1.

Consider the following locally closed, irreducible, subvarieties ($\forall n \in \mathbb{N}$)

$$\begin{aligned} \Delta_{\lambda-\nu} &= \{(X, X') \in \mathbf{E}_\Omega(D, V, V) | X \hookrightarrow X'\}, \quad \dim V = \nu, \\ Y_{\lambda-\nu,\lambda-(\nu+n\alpha_i)} &= \{(X, X') \in \mathbf{E}_\Omega(D, V, V') | X \hookrightarrow X'\}, \quad \dim V = \nu, \dim V' = \nu + ni, \\ Y_{\lambda-\nu,\lambda-(\nu-n\alpha_i)} &= \{(X, X') \in \mathbf{E}_\Omega(D, V, V') | X' \hookrightarrow X\}, \quad \dim V = \nu, \dim V' = \nu - ni, \end{aligned}$$

where the notation $X \hookrightarrow X'$ is defined in Section 4.12. Note that the varieties $\tilde{\Delta}_{\underline{\nu}}$, $\tilde{Y}_{\underline{\nu},-n\alpha_i}$ and $\tilde{Y}_{\underline{\nu},+n\alpha_i}$ are G_D-orbits in $\mathcal{F} \times \mathcal{F}$. Thus their inverse images under u are $\mathbf{G}$-orbits in $\mathbf{E}_\Omega(D, V, V')$, which are open in the varieties $\Delta_{\lambda-\nu}$, $Y_{\lambda-\nu,\lambda-(\nu+n\alpha_i)}$ and $Y_{\lambda-\nu,\lambda-(\nu-n\alpha_i)}$, respectively. From this observation, we have

$$\begin{aligned} \mathscr{I}_\mu &= Q\left(\widetilde{\mathrm{IC}}_{\mathbf{G}}(\overline{\Delta_\mu}) \right), \\ \mathscr{E}^{(n)}_{\mu,\mu-n\alpha_i} &= Q\left(\widetilde{\mathrm{IC}}_{\mathbf{G}}(\overline{Y_{\mu,\mu-n\alpha_i}})[n(\nu_{i+1} - (\nu_i + n))] \right), \\ \mathscr{F}^{(n)}_{\mu,\mu+n\alpha_i} &= Q\left(\widetilde{\mathrm{IC}}_{\mathbf{G}}(\overline{Y_{\mu,\mu+n\alpha_i}})[n((\nu_i - n) - \nu_{i-1})] \right). \end{aligned} \tag{33}$$

Moreover,

Lemma 5.6. *$\mathscr{I}^\Omega_\mu = \mathscr{I}_\mu$, $\mathscr{E}^{(n),\Omega}_{\mu,\mu-n\alpha_i} = \mathscr{E}^{(n)}_{\mu,\mu-n\alpha_i}$, and $\mathscr{F}^{(n),\Omega}_{\mu,\mu-n\alpha_i} = \mathscr{E}^{(n)}_{\mu,\mu-n\alpha_i}$, where $\mathscr{I}^\Omega_\bullet$, $\mathscr{E}^{(n),\Omega}_\bullet$ and $\mathscr{F}^{(n),\Omega}_\bullet$ are defined in Section 4.12.*

Proof. We shall show that $\mathscr{E}^{(n),\Omega}_{\mu,\mu-n\alpha_i} = \mathscr{E}^{(n)}_{\mu,\mu-n\alpha_i}$. The others can be shown in a similar way. Consider the following diagram

$$\begin{array}{ccccc} \mathbf{Z}_{\Omega_i} & \longleftarrow & \hat{\mathbf{Z}} & \xrightarrow{c} & \hat{\mathbf{Z}}_\Omega \\ {\scriptstyle \pi_{12}}\downarrow & & {\scriptstyle \rho}\downarrow & & \downarrow \\ \mathbf{E}_{\Omega_i} & \xleftarrow{\pi} & \mathbf{E}_{\Omega \cup \Omega_i} & \xrightarrow{\pi'} & \mathbf{E}_\Omega, \end{array}$$

where the morphisms π and π' are defined in Section 4.2, π_{12} is defined in Section 4.12, $\hat{\mathbf{Z}} = \mathbf{Z}_{\Omega_i} \times_{\mathbf{E}_\Omega} \mathbf{E}_{\Omega_i \cup \Omega}$, the morphism c is the map by forgetting the components $x_{i-1,i}$ and $x'_{i-1,i}$ and the rest of the morphisms are clearly defined.

By definition, we see that $\mathbf{Z}_{\Omega_i}$ is a locally closed subvariety in the affine variety $\mathbf{E}_\Omega \times \prod_{i \in I} \mathrm{Hom}(V_i, V'_i)$, which, in turn, can be embedded as an open subvariety into a certain variety $\mathbf{E}_\Omega \times \mathbf{P}$, where P is a certain projective variety, by [Z85]. Then the morphism π_{12} is a compactifiable morphism defined in [FK88, p. 86]. From this, and the fact that the left square of the above diagram is cartesian, we may apply the base change for compactifiable morphisms in [FK88, Thm. 8.7] to get the following identity

$$\Phi^\Omega_{\Omega_i}(\pi_{12!}(\widetilde{\mathrm{IC}}_{\mathbf{G}}(\mathbf{Z}_{\Omega_i}))) = \pi'_!(\pi^* \pi_{12!}(\widetilde{\mathrm{IC}}_{\mathbf{G}}(\mathbf{Z}_{\Omega_i})) \otimes \mathcal{L})[r] = \pi'_!(\rho_!(\widetilde{\mathrm{IC}}_{\mathbf{G}}(\hat{\mathbf{Z}})) \otimes \mathcal{L})[r] = \pi'_! \rho_!(\rho^* \mathcal{L})[r],$$

where $r = \nu_{i-1}\nu_i + \nu_{i-1}(\nu_i + n)$.

Let $\hat{\mathbf{Z}}_1$ be the subvariety of $\hat{Z}$ defined by the condition $x_{i-1,i} \overset{\sigma}{\hookrightarrow} x'_{i-1,i}$. Observe that that the map c is a vector bundle with fiber dimension $\nu_{i-1}(\nu_i + n)$ and the restriction of the map

$T\rho$, where T is defined in (6), to the fiber $c^{-1}(X,\sigma)$ of c is 0 if $c^{-1}(X,\sigma) \subseteq \hat{\mathbf{Z}}_1$ and a non constant affine linear function, otherwise. By arguing in exactly the same way as the proof of Proposition 10.2.2 in [L93], we get

$$\Phi^{\Omega}_{\Omega_i}(\pi_{12!}(\widetilde{\mathrm{IC}}_{\mathbf{G}}(\mathbf{Z}_{\Omega_i}))) \simeq \pi^{\Omega}_{12!}(\widetilde{\mathrm{IC}}_{\mathbf{G}}(\mathbf{Z})_{\Omega})[r - 2\nu_{i-1}(\nu_i + n)] = \pi^{\Omega}_{12!}(\widetilde{\mathrm{IC}}_{\mathbf{G}}(\mathbf{Z}_{\Omega}))[-n\nu_{i-1}],$$

where π^{Ω}_{12} is defined similar to π_{12}. Combining the above analysis, we have

$$\begin{aligned}
\mathscr{E}^{(n),\Omega}_{\mu,\mu-n\alpha_i} &= Q\left(\Phi^{\Omega}_{\Omega_i}\pi_{12!}(\widetilde{\mathrm{IC}}_{\mathbf{G}}(\mathbf{Z}_{\Omega_i}))[n(d_i + \sum_{h\in\Omega_i : h'=i}\nu_{h''} - (\nu_i + n))]\right)\\
&= Q\left(\Phi^{\Omega}_{\Omega_i}\pi_{12!}(\widetilde{\mathrm{IC}}_{\mathbf{G}}(\mathbf{Z}_{\Omega_i}))[n(\nu_{i-1} + \nu_{i+1} - (\nu_i + n))]\right)\\
&= Q\left(\pi^{\Omega}_{12!}(\widetilde{\mathrm{IC}}_{\mathbf{G}}(\mathbf{Z}_{\Omega}))[-n\nu_{i-1}][n(\nu_{i-1} + \nu_{i+1} - (\nu_i + n))]\right)\\
&= Q\left(\widetilde{\mathrm{IC}}_{\mathbf{G}}(\overline{Y_{\mu,\mu-n\alpha_i}})[n(\nu_{i+1} - (\nu_i + n))]\right) = \mathscr{E}^{(n)}_{\mu,\mu-n\alpha_i}.
\end{aligned}$$

The lemma follows. □

By (29), (32), Lemmas 5.5, 5.6, and Theorem 5.2, we have

Theorem 5.7. *Under the assumption in this section, the conjectures 4.13 and 4.14 hold. Moreover, the algebra homomorphism Ψ^{Ω}_d in Conjecture 4.14 is the following composition of algebra homomorphisms*

$$_{\mathbb{A}}\dot{\mathbf{U}}(\mathfrak{sl}_{N+1}) \overset{\Psi_d}{\to} (\mathcal{K}_d, \circ) \overset{\bar{Q}Qu^*}{\to} (\mathscr{K}^{\Omega}_d, \cdot),$$

where Ψ_d is in Theorem 5.2 and $\bar{Q}Qu^*$ is an $\mathbb{A}$-algebra isomorphism induced by the functor $\bar{Q}Qu^*$ in (29) such that $\bar{Q}Qu^*(\mathcal{B}_d) = \mathscr{B}^{\Omega}_d$.

Modulo the proof of Proposition 4.11, we have

Corollary 5.8. *The conjectures 4.13 and 4.14 hold for the orientation Ω_a for $a = 2, \cdots n$ in this section. Moreover, we have an algebra isomorphism $\mathscr{K}^{\Omega}_d \to \mathscr{K}^{\Omega_a}_d$ induced from $\Phi^{\Omega_a}_{\Omega}$.*

5.9. **Characterization of the generators in $\mathscr{K}^{\Omega_a}_d$.** We preserve the setting in section 5.3. Let

$$d_j = \nu_j - \nu_{j-1}, \quad \forall j = 1, \cdots, N+1.$$

Since the variety $\tilde{Y}_{\mu,\mu-n\alpha_i}$ in section 5.3 is simply a G_D-orbit, the subvariety $u^{-1}(\tilde{Y}_{\mu,\mu-n\alpha_i})$ is the $\mathbf{G}$-orbit $\mathcal{O}_{M_i}$ in U and furthermore, M_i is the following representation

$$M_i \simeq \oplus_{j=1}^{i} T_j^{\oplus d_j} \oplus T_{i+1}^{d_{i+1}-n} \oplus T^{\oplus n}_{i+1,i'} \oplus \oplus_{j=i+2}^{N+1} T_j^{\oplus d_j}.$$

Therefore, the complex $\widetilde{\mathrm{IC}}_{\mathbf{G}}(\mathcal{O}_{M_i})[n(d_i - n)]$ on $\mathbf{E}_{\Omega}(D, V, V')$ satisfies that the restriction to U is isomorphic to $Qu^*(E^{(n)}_{\mu,\mu-n\alpha_i})$, i.e., $\mathscr{E}^{(n),\Omega}_{\mu,\mu-n\alpha_i} = Q(\widetilde{\mathrm{IC}}_{\mathbf{G}}(\overline{\mathcal{O}_{M_i}})[n(d_{i+1} - n)])$. Since the Fourier-Deligne transform sends a simple perverse sheaf to a simple perverse sheaf, the complex $\mathscr{E}^{(n),\Omega_i}_{\mu,\mu-n\alpha_i}$ is again a simple perverse sheaf, up to a shift. Thus, $\mathscr{E}^{(n),\Omega_i}_{\mu,\mu-n\alpha_i}$ has to be of the form

$$\mathscr{E}^{(n),\Omega_i}_{\mu,\mu-n\alpha_i} = \widetilde{\mathrm{IC}}_{\mathbf{G}}(\overline{\mathcal{O}_{W_i}})[\dim \mathcal{O}_{W_i} - \dim \mathcal{O}_{M_i} + n(d_{i+1} - n)],$$

where $\mathcal{O}_{W_i}$ is certain $\mathbf{G}$-orbit in $\mathbf{E}_{\Omega_i}(D, V, V')$. This section is devoted to describe the orbit $\mathcal{O}_{W_i}$ for $i \geq 2$.

The dimension of the $\mathbf{G}$-orbit $\mathcal{O}_{M_i}$ can be computed as follows. We set

$$\mathbf{i} = (\omega_{N+1}, \omega_N, \cdots, \omega_{i-1}, \tau_i, \omega_i, \cdots, \omega_1)$$

where, for $1 \leq j \leq N+1$,

$$\tau_i = (i', (i+1), (i+1)', \cdots, N, N', N+1), \quad \omega_j = (j, j', (j+1), (j+1)', \cdots, N, N', (N+1)).$$

We also set $\mathbf{a} = (\mathbf{a}_{N+1}, \mathbf{a}_N, \cdots, \mathbf{a}_{i-1}, \check{\mathbf{a}}_i, \mathbf{a}_i, \cdots, \mathbf{a}_1)$ where $\check{\mathbf{a}}_i = (n \cdots, n)$ is a sequence, whose length is the same as τ_i, of the number n, $\mathbf{a}_j$ is a sequence, whose length is the same as ω_j, of the number d_j if $j \neq i+1$, and of the number $d_{i+1} - n$ if $j = i+1$. By [R08], we see that the morphism $\pi_{\mathbf{i},\mathbf{a}} : \widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}} \to \mathbf{E}_\Omega(D, V, V')$, where the pair $(\mathbf{i}, \mathbf{a})$ is just defined above, has image equal to the closure of the orbit $\mathcal{O}_{M_i}$, and the restriction of $\pi_{\mathbf{i},\mathbf{a}}$ to $\pi_{\mathbf{i},\mathbf{a}}^{-1}(\mathcal{O}_{\mathbf{i},\mathbf{a}})$ is an isomorphism, i.e., a resolution of singularity of the orbit closure of $\mathcal{O}_{M_i}$. From this, we see that

$$\dim \mathcal{O}_{M_i} = \dim \widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}} = \sum_{j=1}^{N} \sum_{l'<l: i_l = j+1, i_{l'} = j} a_l a_{l'} + \sum_{j=1}^{N+1} \sum_{l<l': j_l = j = j_{l'}} a_l a_{l'}. \tag{34}$$

We refer to [KS90] for the definition of the *singular support* of a simple perverse sheaf on a complex manifold. When we say the singular support of a simple perverse sheaf K on a smooth algebraic variety X over k of positive characteristic, we mean the singular support of its counterpart on $X(\mathbb{C})$ via the principals in [BBD82, Ch. 6]. Given a simple perverse sheaf K in $\mathcal{D}_G^b(X)$, the singular support, $\mathrm{SS}(K)$, of K is defined to be the singular support of the complex K_X.

Let $\overline{\Omega}$ be the quiver obtained from Ω by reversing all arrows in Ω. We shall identify the space

$$\mathbf{E} := \mathbf{E}_\Omega(D, V, V') \oplus \mathbf{E}_{\overline{\Omega}}(D, V, V')$$

with the cotangent bundles of $\mathbf{E}_\Omega(D, V, V')$ and $\mathbf{E}_{\Omega_i}(D, V, V')$. (See [L91, 12] for more details.) Under such an identification, we have

$$\mathrm{SS}(\mathscr{E}_{\mu,\mu-n\alpha_i}^{(n),\Omega}) = \mathrm{SS}(\mathscr{E}_{\mu,\mu-n\alpha_i}^{(n),\Omega_i}), \tag{35}$$

due to the fact that the singular support of a simple perverse sheaf remains invariant under the Fourier-Sato transform ([KS90]) (a characteristic zero counterpart of Fourier-Deligne transform). By [L91, 13], we have

$$\overline{T^*_{\mathcal{O}_{M_i}} \mathbf{E}_\Omega(D, V, V')} \subseteq \mathrm{SS}(\mathscr{E}_{\mu,\mu-n\alpha_i}^{(n),\Omega}) \subseteq \overline{T^*_{\mathcal{O}_{M_i}} \mathbf{E}_\Omega(D, V, V')} \cup \cup_{\mathcal{O} \subset \overline{\mathcal{O}_{M_i}}} \overline{T^*_{\mathcal{O}} \mathbf{E}_\Omega(D, V, V')}$$

where $\overline{T^*_{\mathcal{O}_{M_i}} \mathbf{E}_\Omega(D, V, V')}$ is the closure of the conormal bundle to $\mathcal{O}_{M_i}$ in $\mathbf{E}_\Omega(D, V, V')$, and $\mathcal{O}$ runs over all $\mathbf{G}$-orbits in the orbit closure $\overline{\mathcal{O}_{M_i}}$. Similarly, we have

$$\overline{T^*_{\mathcal{O}_{W_i}} \mathbf{E}_{\Omega_i}(D, V, V')} \subseteq \mathrm{SS}(\mathscr{E}_{\mu,\mu-n\alpha_i}^{(n),\Omega_i}) \subseteq \overline{T^*_{\mathcal{O}_{W_i}} \mathbf{E}_{\Omega_i}(D, V, V')} \cup \cup_{\mathcal{O} \subset \overline{\mathcal{O}_{W_i}}} \overline{T^*_{\mathcal{O}} \mathbf{E}_{\Omega_i}(D, V, V')}$$

By [KS97, Thm. 6.2.2], $\overline{T^*_{\mathcal{O}_{M_i}} \mathbf{E}_\Omega(D, V, V')}$ and $\overline{T^*_{\mathcal{O}_{W_i}} \mathbf{E}_{\Omega_i}(D, V, V')}$, which are closed, conic, Lagrangian subvarieties of $\mathbf{E}$, are the leading terms in $\mathrm{SS}(\mathscr{E}_{\mu,\mu-n\alpha_i}^{(n),\Omega})$ and $\mathrm{SS}(\mathscr{E}_{\mu,\mu-n\alpha_i}^{(n),\Omega_i})$, respectively. By (35), we have $\overline{T^*_{\mathcal{O}_{M_i}} \mathbf{E}_\Omega(D, V, V')} = \overline{T^*_{\mathcal{O}_{W_i}} \mathbf{E}_{\Omega_i}(D, V, V')}$. This implies that

$$T^*_{\mathcal{O}_{M_i}} \mathbf{E}_\Omega(D, V, V') \cap T^*_{\mathcal{O}_{W_i}} \mathbf{E}_{\Omega_i}(D, V, V') \neq \emptyset,$$

and open dense in $\overline{T^*_{\mathcal{O}_{M_i}} \mathbf{E}_\Omega(D, V, V')}$. So the orbit $\mathcal{O}_{W_i}$ is the orbit with the largest dimension among the $\mathbf{G}$-orbits in the projection of $T^*_{\mathcal{O}_{M_i}} \mathbf{E}_\Omega(D, V, V')$ to $\mathbf{E}_{\Omega_i}(D, V, V')$, i.e.,

$$\overline{p_{\Omega_i}(T^*_{\mathcal{O}_{M_i}} \mathbf{E}_\Omega(D, V, V'))} = \overline{\mathcal{O}_{W_i}}, \tag{36}$$

where $p_{\Omega_i} : \mathbf{E} \to \mathbf{E}_{\Omega_i}(D, V, V')$ is the projection map.

Let Λ be the closed subvariety of $\mathbf{E}$ defined by the following (ADHM or GP) relations:

$$x_{a,a+1}x_{a+1,a} = x_{a,a-1}x_{a-1,a}, \quad x'_{a',(a+1)'}x'_{(a+1)',a'} = x'_{a',(a-1)'}x'_{(a-1)',a'}, \quad \forall 1 \leq a \leq N;$$

$$x_{N+1,N}x_{N,N+1} = x'_{N+1,N'}x'_{N',N+1}.$$

This is nothing but Lusztig's quiver variety in [L91, 12]. By [L91, 12.8], we have

$$x \in T^*_{\mathcal{O}_{M_i}} \mathbf{E}_\Omega(D, V, V') \text{ if and only if } x \in \Lambda \text{ and } x_\Omega \in \mathcal{O}_{M_i}, \tag{37}$$

where x_Ω is the restriction of x to the component $\mathbf{E}_\Omega(D, V, V')$. Now a representative M_i of elements in $\mathcal{O}_{M_i}$ is of the following form $x_{j+1,j} = \binom{I_{\nu_j}}{0}$ and $x_{(j+1)',j'} = \binom{I_{\nu_{j'}}}{0}$, for $j = 0, \cdots, N$, where I_a is the identity matrix of rank a and 0 is the zero matrix and its dimension is determined by the place where it stands. Then, it is not hard to see that an element x in Λ such that $x_\Omega = M_i$ must satisfy that the components $x_{j,j+1}$ are the strict upper triangular matrices of the form

$$\mathbf{A}_j := \begin{pmatrix} 0 & A_{12} & A_{13} & \cdots & A_{1j} & A_{1,j+1} \\ 0 & 0 & A_{23} & \cdots & A_{2j} & A_{2,j+1} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & A_{j,j+1} \end{pmatrix}$$

with the dimension of A_{kl} being $d_k \times d_l$, and the components $x'_{j',(j+1)'}$ are equal to $\mathbf{A}_j$ if $j \leq i-2$ or $j > i$, and is of the form $\mathbf{B}_{i-1} = (\mathbf{A}_{i-1}, \star)$ for $j = i-1$ satisfies certain conditions. From this observation, (36) and (37), we see that the orbit $\mathcal{O}_{W_i}$ is the one with the largest dimension among the $\mathbf{G}$-orbits in $\mathbf{E}_{\Omega_i}(D, V, V')$ having a representative, x_{Ω_i}, whose components at $\Omega_i \backslash \{i \to i-1, i' \to (i-1)'\}$ are equal to the corresponding components in x_Ω and equal to $\mathbf{A}_{i-1}$ and $\mathbf{B}_{i-1}$ on the components $i \to i-1$ and $i' \to (i-1)'$, respectively.

Now, consider the stabilizer, $\mathrm{Stab}_{\mathbf{G}}(x_{\Omega_i})$, of the element x_{Ω_i} in $\mathbf{G}$. It is then straightforward to check that the stabilizer $\mathrm{Stab}_{\mathbf{G}}(x_{\Omega_i})$ consists of all tuples $(g_{i-1}, g_i; h_{i-1}, h_i; g_N)$ in $G_{V_{i-1}} \times G_{V_i} \times G_{V'_{i-1}} \times G_{V'_i} \times G_D$ where g_{i-1} and h_{i-1} are upper triangular block matrices with the block sizes in the diagonal equal to $d_1, d_2, \cdots, d_{i-1}$, h_i is of the form $h_i = \begin{pmatrix} g_i & \star \\ 0 & \star \end{pmatrix}$, and g_N is of the form $g_N = \begin{pmatrix} h_i & \star \\ 0 & g'_N \end{pmatrix}$, with g'_N an upper triangular matrices with block sizes $d_{i+1} - n$, d_{i+2}, $\cdots$, d_N, such that $\mathbf{A}_{i-1} g_i = g_{i-1} \mathbf{A}_{i-1}$ and $\mathbf{B}_{i-1} h_i = h_{i-1} \mathbf{B}_{i-1}$.

If we take $(g_{i-1}, g_i; h_{i-1}, h_i)$ to be diagonal block matrices, we can show that x_{Ω_i} is in the same orbit as certain element y_{Ω_i} of the same form , satisfying an extra condition that, for $j = 1, \cdots, i-1$, the blocks $A_{j,j+1}$ are of the form $A_{j,j+1} = \begin{pmatrix} I_{r_j} & 0 \\ 0 & 0 \end{pmatrix}$ where r_j is the rank of $A_{j,j+1}$. Summing up, we have

Lemma 5.10. *The orbit $\mathcal{O}_{W_i}$ is the one with the largest dimension among the orbits having a presentative of the form x_{Ω_i} such that $A_{j,j+1} = \begin{pmatrix} I_{r_j} & 0 \\ 0 & 0 \end{pmatrix}$ for $j = 1, \cdots, i-1$.*

An immediate consequence of Lemma 5.10 is that, when $N = 2$ and $i = 2$, the orbit $\mathcal{O}_{W_i}$ is exactly the one such that $\mathbf{B}_1$ has the largest rank. After some lengthy calculation and taking into account of (34), we have

$$\mathscr{E}^{(n),\Omega_2}_{\mu,\mu-n\alpha_2} = \begin{cases} \widetilde{\mathrm{IC}}_{\mathbf{G}}(\overline{\mathcal{O}_{W_2}})[n(d+\nu_1-(\nu_2+n))], & \text{if} \quad 2\nu_1 \le \nu_2; \\ \widetilde{\mathrm{IC}}_{\mathbf{G}}(\overline{\mathcal{O}_{W_2}})[n(d+\nu_1-(\nu_2+n))+2\nu_1(\nu_2-2\nu_1)], & \text{if} \quad 2\nu_1 > \nu_2; \end{cases} \tag{38}$$

where

$$W_2 = \begin{cases} T_1^{\oplus d_1} \oplus T_2^{\oplus d_2} \oplus T_3^{\oplus d_3 - n} \oplus S_{2',3}^{\oplus n} & \text{if} \quad 2\nu_1 \le \nu_2; \\ T_1^{\oplus d_2} \oplus T_2^{\oplus d_1} \oplus T_3^{\oplus d_3 - n} \oplus S_1^{\oplus d_1 - d_2} \oplus S_{1'}^{\oplus d_1 - d_2 - n} \oplus S_{1',3}^{\oplus n}, & \text{if} \quad 2\nu_1 - \nu_2 \ge n; \\ T_1^{\oplus d_2} \oplus T_2^{\oplus d_1} \oplus T_3^{\oplus d_3 - n} \oplus S_1^{\oplus d_1 - d_2} \oplus S_{2',3}^{\oplus n-(d_1-d_2)} \oplus S_{1',3}^{\oplus d_1 - d_2}, & \text{if} \quad 2\nu_1 - \nu_2 < n. \end{cases}$$

The map $(X, X') \mapsto (X', X)$ defines an isomorphism $Y_{\mu,\mu-n\alpha_i} \simeq Y_{\mu',\mu'+n\alpha_i}$ where $\mu' = \lambda - (\nu + n\alpha_i)$. The following description of $\mathscr{F}^{(n),\Omega_2}_{\mu',\mu'+n\alpha_i}$ is a direct consequence of (38).

$$\mathscr{F}^{(n),\Omega_2}_{\mu',\mu'+n\alpha_i} = \begin{cases} \widetilde{\mathrm{IC}}_{\mathbf{G}}(\overline{\mathcal{O}_{W_2^t}})[n(\nu_2+n)], & \text{if} \quad 2\nu_1 \le \nu_2; \\ \widetilde{\mathrm{IC}}_{\mathbf{G}}(\overline{\mathcal{O}_{W_2^t}})[n(\nu_2+n)+2\nu_1(\nu_2-2\nu_1)], & \text{if} \quad 2\nu_1 > \nu_2; \end{cases} \tag{39}$$

where

$$W_2^t = \begin{cases} T_1^{\oplus d_1} \oplus T_2^{\oplus d_2} \oplus T_3^{\oplus d_3 - n} \oplus S_{2,3}^{\oplus n}, & \text{if} \quad 2\nu_1 \le \nu_2; \\ T_1^{\oplus d_2} \oplus T_2^{\oplus d_1} \oplus T_3^{\oplus d_3 - n} \oplus S_{1'}^{\oplus d_1 - d_2} \oplus S_1^{\oplus d_1 - d_2 - n} \oplus S_{1,3}^{\oplus n}, & \text{if} \quad 2\nu_1 - \nu_2 \ge n; \\ T_1^{\oplus d_2} \oplus T_2^{\oplus d_1} \oplus T_3^{\oplus d_3 - n} \oplus S_{1'}^{\oplus d_1 - d_2} \oplus S_{2,3}^{\oplus n-(d_1-d_2)} \oplus S_{1,3}^{\oplus d_1 - d_2}, & \text{if} \quad 0 < 2\nu_1 - \nu_2 < n. \end{cases}$$

A similar calculation to that of $\mathcal{E}^{(n),\Omega_2}_{\mu,\mu-n\alpha_i}$ shows that

$$\mathscr{I}^{\Omega_2}_{\mu} = \begin{cases} \widetilde{\mathrm{IC}}_{\mathbf{G}}(\overline{\mathcal{O}_{W_2}}), & \text{if} \quad 2\nu_1 \le \nu_2; \\ \widetilde{\mathrm{IC}}_{\mathbf{G}}(\overline{\mathcal{O}_{W_2}})[2\nu_2(\nu_2-2\nu_1)], & \text{if} \quad 2\nu_1 > \nu_2; \end{cases} \tag{40}$$

where

$$W_2 = \begin{cases} T_1^{\oplus d_1} \oplus T_2^{\oplus d_2} \oplus T_3^{\oplus d_3}, & \text{if} \quad 2\nu_1 \le \nu_2; \\ T_1^{\oplus d_2} \oplus T_2^{\oplus d_1} \oplus T_3^{\oplus d_3} \oplus S_1^{\oplus d_1 - d_2} \oplus S_{1'}^{\oplus d_1 - d_2}, & \text{if} \quad 2\nu_1 > \nu_2. \end{cases}$$

5.11. **A characterization of $\mathscr{Q}^{\Omega}_d$.** We preserve the setting in the sections 5.3 and 5.9. Recall from the section 5.3 that we obtain a semisimple complex $L^{\Omega}_{\mathbf{i},\mathbf{a}} := \pi^{\Omega}_{\mathbf{i},\mathbf{a}}(\bar{\mathbb{Q}}_{l,\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}}})$ on $\mathbf{E}_{\Omega}(D, V, V')$ for any pair $(\mathbf{i}, \mathbf{a})$. Let $\mathcal{Q}^{\Omega}_{d,\nu,\nu'}$ be the full subcategory of $\mathcal{D}^b_{\mathbf{G}}(\mathbf{E}_{\Omega}(D, V, V'))$ consisting of all semisimple complexes such that the simple components, up to a shift, are direct summands of the complexes $L^{\Omega}_{\mathbf{i},\mathbf{a}}$ for various $(\mathbf{i}, \mathbf{a})$.

Let Nil be the variety of nilpotent elements in $\mathrm{End}(D)$. Let G_D acts on Nil by conjugation. Define a morphism

$$\pi : \Lambda \to \mathrm{Nil}$$

of varieties, where Λ is defined in the section 5.9, by sending elements, say X, in Λ to $x_{N+1,N}x_{N,N+1}$. Note that the morphism π is $\mathbf{G}_1$-invariant.

Let Λ^s be the open subvariety of Λ consisting of all elements such that $x_{a+1,a} + x_{a-1,a}$ and $x_{(a+1)',a'} + x_{(a-1)',a'}$ are injective for $1 \leq a \leq N$. By [N94], the $\mathbf{G}_1$-action on Λ^s is free and, moreover, its quotient $\mathbf{G}_1 \backslash \Lambda^s$ is isomorphic to the generalized Steinberg variety

$$\mathcal{Z} := \{(x, F, F') \in \mathrm{Nil} \times \mathcal{F}_{\underline{\nu}} \times \mathcal{F}_{\underline{\nu}'} | x(F_a) \subseteq F_{a-1},\ x(F'_a) \subseteq F'_{a-1},\ 1 \leq a \leq N+1\}.$$

We shall identify $\mathbf{G}_1 \backslash \Lambda^s$ with $\mathcal{Z}$. Due to the fact that π is $\mathbf{G}_1$-invariant, it factors through $\mathcal{Z}$, i.e., π is the composition of the morphisms $\Lambda^s \xrightarrow{q} \mathcal{Z} \xrightarrow{\pi'} \mathrm{Nil}$.

Proposition 5.12. *We have $K \in \mathcal{Q}^{\Omega}_{d,\nu,\nu'} \cap \mathcal{N}$ if and only if $SS(K) \cap \Lambda^s = \emptyset$. Moreover, the restriction of the localization functor Q to $\mathcal{Q}^{\Omega}_{d,\nu,\nu'}$ defines a functor $Q : \mathcal{Q}^{\Omega}_{d,\nu,\nu'} \to \mathscr{Q}^{\Omega}_{d,\nu,\nu'}$ where $\mathscr{Q}^{\Omega}_{d,\nu,\nu'}$ is the full subcategory of $\mathscr{D}^b_{\mathbf{G}}(\mathbf{E}_{\Omega}(D, V, V'))$ consisting of all complexes of the form $\bar{Q}Qu^*(K)$ for $K \in \mathcal{Q}_{d,\underline{\nu},\underline{\nu}'}$ in Section 5.1.*

Proof. Suppose that $\mathrm{SS}(K) \cap \Lambda^s = \emptyset$. From [D04], $\mathrm{SS}(K) \cap \mathbf{E}_{\Omega}(D, V, V') \simeq \mathrm{supp}(K)$ where $\mathbf{E}_{\Omega}(D, V, V')$ is identified with the subspace $0 \oplus \mathbf{E}_{\Omega}(D, V, V')$ of $\mathbf{E}$. So we have

$$\emptyset = \mathrm{SS}(K) \cap \Lambda^s = \mathrm{SS}(K) \cap \Lambda^s \cap \mathbf{E}_{\Omega}(D, V, V') = \mathrm{Supp}(K) \cap U$$

where U is the open subvariety of $\mathbf{E}_{\Omega}(D, V, V')$ defined in the section 5.3. By Lemma 5.4, we see that $K \in \mathcal{Q}^{\Omega}_{d,\nu,\nu'} \cap \mathcal{N}$.

On the other hand, if $\mathrm{SS}(K) \cap \Lambda^s \neq \emptyset$, then it is a non empty closed subvariety of Λ^s. Hence, $\pi(\mathrm{SS}(K) \cap \Lambda^s)$ is a closed subvariety of Nil due to the fact that π can be decomposed as $\pi' q$ with π' a proper map and q a quotient map. Note that $\pi(\mathrm{SS}(K) \cap \Lambda^s)$ is also G_D-invariant since K is a $\mathbf{G}$-equivariant complex. So $0 \in \pi(\mathrm{SS}(K) \cap \Lambda^s)$, which implies that $\mathrm{SS}(K) \cap \Lambda^s \cap \pi^{-1}(0) \neq \emptyset$. Observe that $\Lambda^s \cap \pi^{-1}(0) = U$. Thus, $\mathrm{Supp}(K) \cap U \neq \emptyset$. By Lemma 5.4, this implies that if $\mathrm{SS}(K) \cap \Lambda^s = \emptyset$ then $K \in \mathcal{Q}^{\Omega}_{d,\nu,\nu'} \cap \mathcal{N}$.

It is clear that for any $\mathbf{G}$-equivariant simple perverse sheaf over $\mathbf{E}_{\Omega}(D, V, V')$, if $\mathrm{supp}(K) \cap U \neq \emptyset$, then K comes from certain element in $\mathcal{Q}_{d,\underline{\nu},\underline{\nu}'}$. So $Q(K)$ has to be in $\mathscr{Q}^{\Omega}_{d,\nu,\nu'}$. The proposition follows. $\square$

6. Remarks

(1). Somehow, we are trying to complete the following picture. Let $\mathbf{E}^2$ and Λ^2 be the double of the two varieties defined by Lusztig ([L91]). Let $\Lambda^{2,s}$ be the open subvariety of stable pairs such that the quotient $\Lambda^{2,s}$ by the group $\mathbf{G}$ is Nakajima's quiver variety $\mathbf{Z}$ (of triple). We then have the following diagram

$$\begin{tikzcd}
\mathcal{D}_{\mathbf{G}}(\mathbf{E}^2) \arrow[r] \arrow[d] & \mathcal{D}^b(\mathrm{coh}_A(\Lambda^2)) \arrow[d, "j^*"'] & \\
\mathscr{D}_{\mathbf{G}}(\mathbf{E}^2) \arrow[r, "\Theta"] & \mathcal{D}^b(\mathrm{coh}_A(\Lambda^{2,s})) \arrow[r, "\pi_\flat"] & \mathcal{D}^b(\mathrm{coh}_A(\mathbf{Z})),
\end{tikzcd}$$

where $\mathcal{D}_{\mathbf{G}}(\mathbf{E}^2)$ should be understood as the derived category $\mathrm{MHM}_{\mathbf{G}}(\mathbf{E}^2)$ of mixed hodge modules and $\mathrm{coh}_A(\Lambda^2)$ is the A-equivariant coherent sheaves over Λ^2 with A a certain linear algebraic group.

The top horizontal morphism is constructed by Grojnowski ([G94]), while the bottom horizontal morphism is still missing.

It should be true that the functor $\pi_\flat \circ \Theta$ gives rise to an algebra homomorphism on the Grothendieck group level. This is the connection of the construction in this paper with that of Nakajima ([N00]).

(2). A categorification of the q-Schur-algebra using categories of Harish-Chandra bimodules with the ordinary tensor product of bimodules was obtained in [MS08, Theorem 47]. It should be directly connected to our setup via localisation, as was pointed out by Stroppel.

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